Maximum and Minimum Values

Concepts: Suppose that $f(x)$ is continuous on an interval I and c is a point inside I.
(i) If $f(x) \leq f(c)$ (or $f(x) \geq f(c)$, respectively) for all x in I, then $f(c)$ is an **absolute maximum** (or an **absolute minimum**, respectively) of $f(x)$ on I. An absolute maximum or an absolute minimum of f is an **absolute extremum** of f.
(ii) If $f(x) \leq f(c)$ (or $f(x) \geq f(c)$, respectively) for all x in some **open** interval containing c, then $f(c)$ is a **local maximum** (or a **local minimum**, respectively) of $f(x)$. A local maximum or a local minimum of f is a **local extremum** of f.
(iii) A number c in the domain of f is a **critical point** of f if **either** $f'(c) = 0$ **or** $f'(c)$ does not exist.
(iv) (**How to find extrema of a function?**) Suppose that f is continuous on $[a, b]$. Then the absolute extrema of f on $[a, b]$ must occur at either a critical point inside (a, b), or at an end point of the interval ($x = a$ or $x = b$).

Example 1 Find all critical points of $f(x) = x^3 - 3x^2 + 3x$, and determine whether each represents a local maximum, local minimum, or neither. (Page 267, #9)

Solution: Since $f(x)$ is a polynomial, $f(x)$ is differentiable (and so continuous) in its domain $(-\infty, \infty)$.

Compute

$$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2.$$

Thus the only critical point of $f(x)$ is $x = 1$, which divides the domain of $f(x)$ into two intervals: $(-\infty, 1)$ and $(1, \infty)$.

Since $f'(x) > 0$, when $x \neq 1$, we conclude that $f'(x) > 0$ in both $(-\infty, 1)$ and $(1, \infty)$. Thus $y = f(x)$ is increasing in both intervals. Therefore, $f(1)$ is neither a local maximum value nor a local minimum value of $f(x)$ in its domain.

Example 2 Find all critical points of $f(x) = x^{\frac{3}{4}} - 4x^{\frac{1}{4}}$, and determine whether each represents a local maximum, local minimum, or neither. (Page 267, #13)

Solution: Since $f(x)$ is a linear combination of power functions, $f(x)$ is continuous on its domain $[0, \infty)$ and differentiable in $(0, \infty)$.

Compute

$$f'(x) = \frac{3}{4}x^{-\frac{1}{4}} - x^{-\frac{3}{4}} = \frac{3x^{\frac{2}{4}} - 4}{4x^{\frac{3}{4}}}.$$

As $x = 0$ is in the domain of $f(x)$ and $f'(0)$ does not exist, thus $x = 0$ is a critical point. Set $f'(x) = 0$ to get $x^{\frac{2}{4}} = \frac{4}{3}$, or $x = \frac{16}{9}$. Hence the critical points of $f(x)$ are $x = 0$ and $x = \frac{16}{9}$, which divide the domain of $f(x)$ into two intervals: $(0, \frac{16}{9})$ and $(\frac{16}{9}, \infty)$.

Since $f'(1) < 0$ and $f'(16) > 0$, we conclude that $f'(x) < 0$ in $(0, \frac{16}{9})$ and $f'(x) > 0$ in $(\frac{16}{9}, \infty)$. Thus $f(0)$ is a local maximum value, and $f(\frac{16}{9})$ is a local minimum value of $f(x)$ in its domain.

Example 3 Find all critical points of $f(x) = \sin x \cos x$ on $[0, 2\pi]$, and determine whether each represents a local maximum, local minimum, or neither. (Page 267, #17)

Solution: Recall that $f'(x)$ exists in $(0, 2\pi)$. Compute

$$f'(x) = \underline{\cos x} \cos x + \sin x \underline{(-\sin x)} = \cos^2 x - \sin^2 x.$$

Thus the critical points of $f(x)$ are solutions of $\cos^2 x - \sin^2 x$, or $\tan^2 x = 1$, inside $(0, 2\pi)$. We have solutions $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. These points partition the interval $(0, 2\pi)$ into 5 intervals.

$$(0, \frac{\pi}{4}), (\frac{\pi}{4}, \frac{3\pi}{4}), (\frac{3\pi}{4}, \frac{5\pi}{4}), (\frac{5\pi}{4}, \frac{7\pi}{4}), \text{ and } (\frac{7\pi}{4}, 2\pi).$$

Since

$$\begin{aligned}
f'(\frac{\pi}{6}) &= \cos^2 \frac{\pi}{6} - \sin^2 \frac{\pi}{6} = \frac{3}{4} - \frac{1}{4} > 0 \\
f'(\frac{\pi}{2}) &= \cos^2 \frac{\pi}{2} - \sin^2 \frac{\pi}{2} = 0 - 1 < 0 \\
f'(\pi) &= \cos^2 \pi - \sin^2 \pi = 1 - 0 > 0 \\
f'(\frac{11\pi}{6}) &= \cos^2 \frac{11\pi}{6} - \sin^2 \frac{11\pi}{6} = \frac{3}{4} - \frac{1}{4} > 0,
\end{aligned}$$

we conclude that $f'(x) > 0$ in $(0, \frac{\pi}{4})$ and in $(\frac{3\pi}{4}, \frac{5\pi}{4})$, and $f'(x) < 0$ in $(\frac{\pi}{4}, \frac{3\pi}{4})$ and in $(\frac{7\pi}{4}, 2\pi)$. Thus $f(\frac{\pi}{4})$ and $f(\frac{5\pi}{4})$ are local maximum values, and $f(\frac{3\pi}{4})$ and $f(\frac{7\pi}{4})$ are local minimum values of $f(x)$ in the given interval.

Example 4 Find the absolute extrema of $f(x) = x^4 - 8x^2 + 2$ on the interval $[-3, 1]$. (Page 268, #35)

Solution: **To find absolute extrema of a function on a closed interval $[a, b]$, we can follow these two steps.**

(Step 1) Find critical points in side $[a, b]$. In this problem, $a = -3$ and $b = 1$.

Note that $f'(x)$ exists in $(-3, 1)$. Compute

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2).$$

Set $f'(x) = 0$ to get critical points $x = 0, x = -2$ and $x = 2$. Only $x = 0$ and $x = -2$ are inside the interval $(-3, 1)$.

(Step 2) Compare the values of $f(x)$ at the critical points inside the interval and the ends of the interval to find out the extrema.

Compute

$$\begin{aligned}
f(-3) &= (-3)^4 - 8 - 3^2 + 2 = 81 - 72 + 2 = 11 \\
f(-2) &= (-2)^4 - 8(-2)^2 + 2 = 16 - 32 + 2 = -14 \\
f(0) &= 2 \\
f(1) &= 1 - 8 + 2 = -5
\end{aligned}$$

By comparison, $f(-3) = 11$ is the absolute maximum value and $f(-2) = -14$ is the absolute minimum value of $f(x)$ on $[-3, 1]$.

Example 5 Find the absolute extrema of $f(x) = \sin x + \cos x$ on the interval $[0, 2\pi]$. (Page 268, #39)

Solution: **To find absolute extrema of a function on a closed interval** $[a, b]$**, we can follow these two steps.**

(Step 1) Find critical points in side $[a, b]$. In this problem, $a = -3$ and $b = 1$.

Note that $f'(x)$ exists in $[0, 2\pi]$. Compute

$$f'(x) = \cos x - \sin x$$

Set $f'(x) = 0$ to get critical points $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. Both are inside the interval $(-3, 1)$.

(Step 2) Compare the values of $f(x)$ at the critical points inside the interval and the ends of the interval to find out the extrema.

Compute

$$\begin{aligned} f(\frac{\pi}{4}) &= \sin(\frac{\pi}{4}) + \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \\ f(\frac{5\pi}{4}) &= \sin(5\frac{\pi}{4}) + \cos(\frac{5\pi}{4}) = \frac{-\sqrt{2}}{2} + \frac{-\sqrt{2}}{2} = -\sqrt{2} \\ f(0) &= \sin(0) + \cos(0) = 0 + 1 = 1 \\ f(2\pi) &= \sin(2\pi) + \cos(2\pi) = 0 + 1 = 1. \end{aligned}$$

By comparison, $f(\frac{\pi}{4}) = \sqrt{2}$ is the absolute maximum value and $f(\frac{5\pi}{4}) = -\sqrt{2}$ is the absolute minimum value of $f(x)$ on $[0, 2\pi]$.

Example 6 Find x to minimize the $(x^2 + 1) - \ln x$. At this value of x, show that the tangent line to $y = f(x) = x^2 + 1$ and $y = g(x) = \ln x$ are parallel. (Page 268, #55)

Solution: Let $F(x) = x^2 + 1 - \ln x$. Then the domain of $F(x)$ is $(0, \infty)$. Compute

$$F'(x) = 2x - \frac{1}{x}.$$

Any extrema must occur at a critical point. Set $F'(x) = 0$ to get critical points $x = \pm\frac{1}{\sqrt{2}}$. Only $x = \frac{1}{\sqrt{2}}$ is in the domain of $F(x)$, which partitions the domain $(0, \infty)$ of $F(x)$ into two intervals

$$(0, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \infty).$$

Since $F'(\frac{1}{2}) = 1 - 2 < 0$ and $F'(1) = 2 - 1 > 0$, (x) is decreasing in $(0, \frac{1}{\sqrt{2}})$ and increasing in $(\frac{1}{\sqrt{2}}, \infty)$. Therefore, $F(\frac{1}{\sqrt{2}})$ is an absolute minimum of $F(x)$ in its domain.

Since $f'(x) = 2x$ and $g(x) = \frac{1}{x}$, at $x = \frac{1}{\sqrt{2}}$, $f'(\frac{1}{\sqrt{2}}) = \sqrt{2} = g'(\frac{1}{\sqrt{2}})$, and so the two tangent lines have the same slope. Therefore, they are parallel.