The exact Turán number of $F(3,3)$ and all extremal configurations

by John Goldwasser* and Ryan Hansen*

July 10, 2012

Mathematics Subject Classification: 05D05

Abstract

If H is a 3-graph, then $\mathrm{ex}(n;H)$ denotes the maximum number of edges in a 3-graph on n vertices containing no sub-3-graph isomorphic to H. Let $S(n)$ denote the 3-graph on n vertices obtained by partitioning the vertex set into parts of sizes $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ and taking as edges all triples that intersect both parts. Let $s(n)$ denote the number of edges in $S(n)$. Let $F(3,3)$ denote the 3-graph $\{123, 145, 146, 156, 245, 246, 256, 345, 346, 356\}$. We prove that if $n \neq 5$ then $\mathrm{ex}(n; F(3,3)) = s(n)$ and that the unique optimal 3-graph is $S(n)$.

Key words: Turan number, hypergraph, F(3,3)

1 Introduction

In this paper, we generally use the standard notation. An r-graph is a collection of subsets of size r of a finite set V, called vertices. We sometimes identify a 3-graph with its edge set. We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$, $X^{(r)}$ to denote the set of all r-subsets of a set X and $e(H)$ to denote the number of edges in an r-graph H.

*Department of Mathematics, West Virginia University, Morgantown, WV 26506, jgoldwas@math.wvu.edu, rhansen@math.wvu.edu

Given an r-graph H, the Turán number of H, $\mathrm{ex}(n; H)$, is the maximum number of edges in an r-graph on n vertices that does not contain a copy of H (we say such a graph is H-free). A simple averaging argument shows that $\mathrm{ex}(n; H)/\binom{n}{r}$ is a non-increasing function of n, so its limit as n goes to infinity exists, is denoted $\pi(H)$, and is called the Turán density of H. It is well known (see, for example, [4]) that if H is an ordinary graph ($r = 2$) then $\pi(H)$ depends only on the chromatic number of H. Turán's theorem [4] gives the exact value of $\mathrm{ex}(n; K_m)$ and the unique K_m-free graph with the maximum number of edges.

Much less is known when $r > 2$. For both $K_4^{(3)} = \{123, 124, 134, 234\}$ and $K_4^- = \{123, 124, 134\}$ there are simple constructions which provide a lower bound ([1] and [5]) and using Razborov's flag algebra approach there has been recent progress in lowering the best upper bound. However, the precise values of $\pi(K_4^{(3)})$ and $\pi(K_4^-)$ have yet to be determined.

In fact, it is only within the last dozen years or so that an exact non-zero value of $\pi(H)$ has been determined for any r-graph H with $r > 2$. The breakthrough was with the Fano plane $F = \{124, 235, 346, 457, 561, 672, 713\}$. De Caen and Füredi [3] developed a method using linkgraphs (explained in section 2) which reduced much of the problem to a question about edge densities of ordinary multigraphs. They showed that $\pi(F) = \frac{3}{4}$ and the method, with modifications, was later used independently in [8] and [13] to determine, for sufficiently large n, the exact value of $\mathrm{ex}(n; F)$ and to show the optimal 3-graph is unique.

The Turán problem has been completely solved for $F_5 = \{123, 124, 345\}$, a 3-graph which first received attention in a theorem of Bollobás proving a conjecture of Katona, because forbidding F_5 and K_4^- is equivalent to requiring that there do not exist three edges such that one contains the symmetric difference of the other two. Frankl and Füredi [6] later showed that $\pi(F_5) = \frac{2}{9}$ and that, for sufficiently large n, the complete equitripartite graph is the unique F_5-free 3-graph with $\mathrm{ex}(n; F_5)$ edges. Keevash and Mubayi [11] used the de Caen-Füredi method to show the same for all n greater than 32, while Goldwasser [9] sharpened the method and found, for all n, all F_5-free graphs with $\mathrm{ex}(n; F_5)$ edges (there is a unique one for all $n \geq 5$, except there are two for $n = 10$).

If p and q are positive integers Mubayi and Rödl [14] defined $F(p, q)$ to be the 3-graph on $p+q$ vertices whose edges are all the 3-sets which intersect a fixed p-set of vertices in 1 or 3 points. They showed $\pi(H) = \frac{3}{4}$ for several 3-graphs in H in $F(p, q)$, or related to a 3-graph in $F(p, q)$, for certain small values of p and q. Among these is $F(3,3)$, the 3-graph on [6] with 10 edges, the "special" edge 123, and the nine 3-subsets of [6] which intersect 123 in precisely one point.

In this paper we determine the exact value of $\mathrm{ex}(n; F(3,3))$ and show that, for each n, there is a unique $F(3,3)$-free 3-graph with $\mathrm{ex}(n; F(3,3))$ edges. That makes $F(3,3)$ the second non-trivial 3-graph H (F_5 is the other) such that all H-free 3-graphs with $\mathrm{ex}(n; H)$ edges have been determined for all n (the unique maximum Fano plane-free graph has been determined for sufficiently large n). After a draft of this paper was written we discovered that Keevash and Mubayi, in [12], recently determined $\mathrm{ex}(n; F(3,3))$ for all n, though they did not show the uniqueness of the optimal 3-graph.

One of the main ingredients in our proof is Proposition 2, a sharpened version of a lemma used in [3], [8], and [13]. The sharpened version is needed to prove uniqueness of the optimal configuration.

For $n \geq 3$, let $S(n)$ denote the 3-graph obtained by partitioning $[n]$ into parts of sizes $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ and taking as edges all triples that intersect both parts. Let $s(n)$ denote the number of edges in $S(n)$. It is easy to calculate that

$$s(n) = \left\lfloor \frac{3}{4} \cdot \frac{n}{n-1} \binom{n}{3} \right\rfloor = \left\lfloor \frac{n^2(n-2)}{8} \right\rfloor .$$

The following is the main result of this paper.

Theorem 1. *Let H be an $F(3,3)$-free 3-graph on $n \neq 5$ points. Then $e(H) \leq s(n)$ with equality holding if and only if H is isomorphic to $S(n)$.*

$F(3,3)$ is not 2-colorable, meaning that in any 2-coloring of its vertices there must be a monochromatic edge (further discussion in Section 4). That there is no 3-3 partition of [6] such that each edge intersects both parts follows because $F(3,3)$ has as an edge precisely one of each of the ten pairs consisting of a 3-subset of [6] and its complement. Since $S(n)$ is 2-colorable, clearly it is $F(3,3)$-free.

2 Definitions and Preliminaries

We define a family $\mathscr{G}(n)$ of multigraphs ($r = 2$) on n vertices as follows. A multigraph G with n vertices is in $\mathscr{G}(n)$ if and only if the following are satisfied:

1. For each $x, y \in V(G)$, the multiplicity $\mu_G(x,y) = \mu(x,y)$ of xy is 2,3 or 4.

2. If $M(G)$ is the (ordinary) graph with $V(M(G)) = V(G)$ and $E(M(G)) =$

   $\{xy \in E(G) \mid \mu_G(x,y) = 4\}$ then

   (a) If n is even, then each component of $M(G)$ is a complete equibipartite graph.

   (b) If n is odd, then one component of $M(G)$ is a complete bipartite graph with part sizes differing by 1 (possibly of sizes 0 and 1) and all other components are complete equibipartite.

3. For each x, y in the same partition part of a component of $M(G)$, $\mu_G(x,y) = 2$.

4. For each x, y in different components of $M(G)$, $\mu_G(x,y) = 3$.

It is easy to check that each G in $\mathscr{G}(n)$ has $3\binom{n}{2} + \left\lfloor \frac{n}{2} \right\rfloor$ edges and that each set of 3 vertices spans at most 10 edges. A graph in $\mathscr{G}(n)$ could have as few as $\left\lfloor \frac{n}{2} \right\rfloor$ edges of multiplicity 4 (they would form a maximum matching and all other edges would have multiplicity 3) and as many as $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges of multiplicity 4 (forming $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ with all other edges having multiplicity 2). We let G_n^* denote this latter multigraph (so $M(G_n^*)$ is complete equibipartite). If $p(n)$ denotes the number of partitions of n, it is easy to see that the number of isomorphically distinct multigraphs in $\mathscr{G}(n)$ is $p(k)$ if $n = 2k$ and $\sum_{i=0}^{k} p(i)$ if $n = 2k+1$.

2.1 Main Proposition

Proposition 2. *Let G be a multigraph on $n \geq 3$ vertices where each set of 3 vertices spans at most 10 edges. Then*

$$e(G) \leq 3\binom{n}{2} + \left\lfloor \frac{n}{2} \right\rfloor . \tag{1}$$

Furthermore if $n \geq 5$ then equality holds if and only if $G \in \mathscr{G}(n)$.

We do the proof in two parts, first the inequality, then the characterization of equality.

Proof Of Inequality In Proposition 2

First we prove inequality (1) by induction on n. It is obviously satisfied if $n = 3$. It is also obviously satisfied if no edge has multiplicity greater than 3. So we assume $n > 3$ and $\mu(x, y) = t \geq 4$ for some edge xy of G. For each z in $V(G) \setminus \{x, y\}$, $\mu(x, z) + \mu(y, z) \leq 10 - t$, so if p is the total number of edges incident to x or y (or both) then

$$\begin{aligned} p &\leq (10 - t)(n - 2) + t && (2)\\ &= 10(n - 2) - t(n - 3) \\ &\leq 10(n - 2) - 4(n - 3) && (3)\\ &= 6n - 8. \end{aligned}$$

Hence, by the inductive hypothesis,

$$\begin{aligned} e(G) &\leq 3\binom{n-2}{2} + \left\lfloor \frac{n-2}{2} \right\rfloor + 6n - 8 \\ &= 3\binom{n}{2} + \left\lfloor \frac{n}{2} \right\rfloor . \end{aligned}$$

Characterization Of Equality In Proposition 2

We prove the statement about equality holding in inequality (1) for $n \geq 5$ by induction on n. We can do the base cases, $n = 5$ and $n = 6$, and the inductive step simultaneously.

Assume $n \geq 5$ and equality holds in (1). Then equality must hold in (3), so no multiplicity can be more than 4. Let x and y be vertices such that $\mu(x, y) = 4$. Since equality holds in (1) and (2), for each $z \in V(G) \setminus \{x, y\}$, both $\mu(x, z)$ and $\mu(y, z)$ are equal to 3 or one is equal to 4 and the other is equal to 2. Moreover, the multigraph $G' = G \setminus \{x, y\}$ has precisely $3\binom{n-2}{2} + \left\lfloor \frac{n-2}{2} \right\rfloor$ edges.

If $n = 5$, then G' has 10 edges, the multiplicities must be 4,4,2 or 4,3,3 and in either case $G' \in \mathscr{G}(3)$. If $n = 6$, then G' has 20 edges, the multiplicities must be 4,4,4,4,2,2 or 4,4,3,3,3,3 (each pair of disjoint edges have the same multiplicities) and, in either case, $G' \in \mathscr{G}(4)$. To complete the proof of both the base cases and the inductive step, it suffices to show that if $n \geq 5, G' \in \mathscr{G}(n - 2)$, and equality holds, then $G \in \mathscr{G}(n)$.

If $\mu(x, z) = \mu(y, z) = 3$ for all $z \in V(G')$, then G is clearly in $\mathscr{G}(n)$ ($M(G)$ has one more component than $M(G')$ and that is a single edge).

Suppose for some $z \in V(G')$, $\mu(x,z) = 2$ and $\mu(y,z) = 4$. Let A, B be the bipartition of the component of $M(G')$ containing z with $z \in A$. For each $v \in B, \mu(z,v) = 4$, so, since $\mu(y,z) = 4$, it follows that $\mu(y,v) = 2$ and $\mu(x,v) = 4$. Then it follows that for each $u \in A$, $\mu(y,u) = 4$ and $\mu(x,u) = 2$. Hence, $G \in \mathscr{G}(n)$ ($M(G)$ has the same components as $M(G')$ except one has two more vertices and bipartition $A \cup \{x\}, B \cup \{y\}$). ■

If $n = 3$, then equality obviously holds in (1) for any nonnegative edge multiplicities a, b, c whose sum is 10. It is not hard to show that if $n = 4$, with vertices w, x, y, z, then equality holds if and only if $u(wx) = u(yz) = a$, $u(wy) = u(xz) = b$, and $u(wz = u(wy) = c)$, for any nonnegative integers a, b, c whose sum is 10. However, if n equals 3 or 4, and no edge multiplicity is greater than 4, then it is not hard to see that equality can hold in (1) only if G is in $\mathscr{G}(n)$.

Comments About Proposition 2

The lemma used in proofs in [3], [8] and [13] has $3\binom{n}{2}+n-2$ in the inequality. The sharp inequality with $3\binom{n}{2} + \left\lfloor \frac{n}{2} \right\rfloor$ is actually a special case of a much more general result of Füredi and Kundgen [7], but they did not characterize equality. In a paper on a weighted generalization of Turán's theorem, Bondy and Tuza [2, Theorem 5.1] actually did characterize equality for an inequality which is a generalization of inequality (1).

2.2 Other Preliminary Results

Since $F(3,3)$ has precisely one edge from each of the ten pairs of a 3-subset of [6] and its complement, the 3-graph $S(6) = [6]^{(3)} \setminus \{123, 456\}$ has 18 edges and is $F(3,3)$-free. The following lemma, another key ingredient in showing uniqueness of the optimal configuration, says that any $F(3,3)$-free 3-graph on six points which is not a subgraph of $S(6)$ has at most 16 edges.

Lemma 3. *If $H \subseteq [6]^{(3)}$ is $F(3,3)$-free and has as an edge at least one of each pair of a set in $[6]^{(3)}$ and its complement, then H has at most 16 edges.*

Proof. There are six isomorphically distinct 3-graphs on [6] with three edges, no two of which are disjoint: $\{124, 125, 126\}$, $\{124, 134, 234\}$, $\{124, 134, 235\}$, $\{124, 135, 236\}$, $\{124, 125, 135\}$, $\{124, 125, 136\}$. Each is disjoint from the copy of $F(3,3)$ where 123 is the special edge (the other nine edges intersect

123 in one point), so any $H \subseteq [6]^{(3)}$ with at least 17 edges and at least one of each complementary pair of sets in $[6]^{(3)}$ contains an $F(3,3)$ subgraph. ■

If v is a vertex of the 3-graph H, the *linkgraph* $H[v]$ of v in H is the (ordinary) graph $H[v] = \{xy \mid vxy \in H\}$. The following lemma is similar in spirit to lemmas which appeared in [3], [9] and [13].

Lemma 4. *If H is an $F(3,3)$-free 3-graph and $\{abc, abd, acd, bcd\} \subseteq H$ then the multiset union $H(a,b,c,d) = H[a] \cup H[b] \cup H[c] \cup H[d]$ is a multigraph such that each 3 points span at most 10 edges.*

Proof. Suppose x, y and z are vertices which span at least 11 edges of $H(a,b,c,d)$. That means $\{x,y,z\}$ spans three edges of three of $H[a], H[b], H[c]$ and $H[d]$, say the first three. That produces a copy of $F(3,3)$ on $\{a,b,c,x,y,z\}$ (abc is the special edge). ■

3 Proof Of Theorem 1

Proof. If $n > 5$ then adding an edge to $S(n)$ creates a copy of $F(3,3)$, so it suffices to show that if H is an $F(3,3)$-free 3-graph on $n \neq 5$ points with $s(n)$ edges, then H is isomorphic to $S(n)$. This will be shown by induction on n. The statement certainly holds for $n \leq 4$. We note that $S(n)$ is not optimal when $n = 5$, since $S(5)$ has 9 edges, while $K_5^{(3)}$ has 10, but is uniquely optimal when $n = 6$. (Lemma 3 says that if $H \subseteq [6]^{(3)}$ has at least 17 edges and is $F(3,3)$-free then H is a subgraph of $S(6)$)

It is easy to check that a $K_4^{(3)}$-free 3-graph on 5 points has at most 7 edges, so a $K_4^{(3)}$-free graph on $n \geq 5$ points has at most $\frac{7}{10}\binom{n}{3}$ edges. Since $s(n) > \frac{3}{4}\binom{n}{3}$, if H is a 3-graph on $n \geq 5$ points with $s(n)$ edges, then it has a $K_4^{(3)}$ subgraph.

Assume H is an $F(3,3)$-free 3-graph on $n \geq 7$ points with $s(n)$ edges. Let $S = \{a,b,c,d\}$ be a subset of $V(H)$ which induces a copy of $K_4^{(3)}$. For $i \in \{0,1,2,3\}$, let $f_i^H(S) = f_i(S)$ denote the number of edges of H which have precisely i vertices in S.

We are going to apply the inductive hypothesis on $H \setminus S$, so to be able to assume $H \setminus S$ has at most $s(n-4)$ edges when $n = 9$, we need to show that H has less than $s(9)$ edges if $H \setminus S$ is $K_5^{(3)}$. If $H \setminus S = K_5^{(3)}$ then, by

Lemma 3, for each $v \in S$, $H \setminus (S \setminus \{v\})$ has at most 16 edges. That means $f_1(S) \leq 4(16-10) = 24$ and

$$\begin{aligned} e(H) &= f_0(S) + f_1(S) + f_2(S) + f_3(S) \\ &\leq 10 + 24 + 30 + 4 \\ &= 68 < 70 = s(9). \end{aligned}$$

Hence, by our inductive assumption, $f_0(S) \leq s(n-4)$ for all $n \geq 7$.

If T is a subset of size 4 of $V(S(n))$ which spans 4 edges, then $f_0(T) = s(n-4)$, $f_1(T) = 3\binom{n-4}{2} + \lfloor \frac{n-4}{2} \rfloor$, $f_2(T) = 5(n-4)$ and $f_3(T) = 4$. We have $f_0(S) \leq s(n-4) = f_0(T)$ and, by Lemma 4 and Proposition 2, $f_1(S) \leq 3\binom{n-4}{2} + \lfloor \frac{n-4}{2} \rfloor = f_1(T)$. Since $e(H) = s(n)$ and $f_3(S) = f_3(T) = 4$, letting $m = n-4$ we must have

$$f_2^H(S) \geq f_2(T) = 5m. \tag{4}$$

We want to show that every vertex in $H \setminus S$ is in precisely 5 edges which have two vertices in S. Let a, b, c, d, e be the number of vertices in $H \setminus S$ which are in, respectively, precisely 6, precisely 5, precisely 4, precisely 3 and at most 2 edges with the other two vertices in S. Then

$$m = a + b + c + d + e, \tag{5}$$

$$\begin{aligned} 5m \leq f_2(S) &\leq 6a + 5b + 4c + 3d + 2e \\ &= 6m - (b + 2c + 3d + 4e) \end{aligned} \tag{6}$$

and hence

$$a \geq c + 2d + 3e. \tag{7}$$

Let A be the set of vertices in $H \setminus S$ which are in 6 edges with the other two vertices in S. By way of contradiction, suppose $|A| = a > 0$. For each $x \in A$, the subgraph of H spanned by $S \cup \{x\}$ is $K_5^{(3)}$. By Lemma 3, for each $y \notin (S \cup \{x\})$, $S \cup \{x, y\}$ spans at most 16 edges of H, at most 6 containing y. So if y is in j edges with the other two vertices in S, then y can be in at most $6 - j$ edges of the form yuv where $u \in A$ and $v \in S$. That means there are no edges with one vertex in S and the other two in A, and at most $(ab + 2ac + 3ad + 4ae)$ edges with one vertex in S, one in A, and one in $V(H) \setminus (S \cup A)$. Hence

$$f_1(S) \leq ab + 2ac + 3ad + 4ae + 3\binom{m-a}{2} + \left\lfloor \frac{m-a}{2} \right\rfloor \tag{8}$$

where we have used Proposition 2 to get an upper bound for the number of edges with one vertex in S and two in $V(H) \setminus S \cup A$. From (6), (7) and (8) we get

$$\begin{aligned}
f_1(S) + f_2(S) &\le 3\binom{m-a}{2} + \left\lfloor \frac{m-a}{2} \right\rfloor + 6m \\
&\qquad - (b + 2c + 3d + 4e) + a(b + 2c + 3d + 4e) \\
&= 3\binom{m-a}{2} + \left\lfloor \frac{m-a}{2} \right\rfloor + 6m \\
&\qquad + (a-1)\Big[\underbrace{(b+c+d+e)}_{=m-a \text{ by } (5)} + \underbrace{(c+2d+3e)}_{\le a \text{ by } (7)}\Big] \\
&\le 3\binom{m-a}{2} + \frac{m-a}{2} + 6m + (a-1)m \\
&= \frac{m-a}{2}\Big(3m - 3a - 2\Big) + 5m + am \\
&= \frac{m(3m+8)}{2} - \frac{a}{2}\Big(4m - 3a - 2\Big) \\
&\le \left\lfloor \frac{m(3m+8)}{2} \right\rfloor + \frac{1}{2} - \frac{a}{2}\Big[(m-2) + 3(m-a)\Big] \\
&< \left\lfloor \frac{m(3m+8)}{2} \right\rfloor
\end{aligned}$$

The last inequality holds because $m \ge 3$ and $a > 0$. Hence,

$$\begin{aligned}
f_1(S) + f_2(S) &< \left\lfloor \frac{m(3m+8)}{2} \right\rfloor \\
&= 3\binom{m}{2} + \left\lfloor \frac{m}{2} \right\rfloor + 5m \\
&= f_1(T) + f_2(T),
\end{aligned}$$

a contradiction, since $f_0(S) = f_0(T)$ and $f_3(S) = f_3(T)$ implies that $f_1(S) + 2f_2(S) = f_1(T) + f_2(T)$. Thus $a = 0$ and, by inequality (7), $c = d = e = 0$. Hence $m = b$ and $f_1(S) = 3\binom{m}{2} + \lfloor \frac{m}{2} \rfloor$, $f_2(S) = 5m$, and each vertex in $V(H) \setminus S$ is in precisely 5 edges of H which have two vertices in S.

Since $f_1(S) = 3\binom{n-4}{2} + \lfloor \frac{n-4}{2} \rfloor$, the multigraph $H(a,b,c,d)$ must be in $\mathscr{G}(n-4)$ (this follows by Lemma 4 and by Proposition 2 if $n - 4 \ge 5$, but it is also true if $n-4$ is equal to 3 or 4, by the remarks after the proof of Proposition 2, since no edge multiplicity of $H(a,b,c,d)$ can be greater than 4). We want to show $H(a,b,c,d)$ is isomorphic to G^*_{n-4}, that is that $M(H(a,b,c,d))$ has only one component (so it is complete equibipartite).

Let x and y be vertices of $H \setminus S$ such that $\mu(x,y) = 4$ and, by way of contradiction, suppose $M(H(a,b,c,d))$ has more than one component. Then there exists $z \in V(H) \setminus S$ such that $\mu(x,z) = \mu(y,z) = 3$. The subgraph of H induced by $S \cup \{x,z\}$ has 17 edges, so, by Lemma 3, two of the three missing edges of $(S \cup \{x,z\})^{(3)}$ must be complementary, and the only possibility is that one contains x and two vertices of S, while the other contains z and the other two vertices of S, say xab and zcd. Similarly $S \cup \{y,z\}$ has 17 edges, and its two missing complementary edges must each contain precisely two vertices of S. Since one is zcd, the other must be yab. This is a contradiction, because $S \cup \{x,y\}$ induces 18 edges in H, and if xab and yab are the two missing edges, then $S \cup \{x,y\}$ induces a copy of $F(3,3)$. Hence, $M(H(a,b,c,d))$ has only one component.

Let A, B be the vertex partition for $M(H(a,b,c,d))$, which is a complete bipartite graph. Suppose $x, z \in A$ and $y \in B$ and that xab and ycd are the missing edges in the subgraph of H induced by $S \cup \{x,y\}$. Hence zab and ycd must be the missing edges in the subgraph of H induced by $S \cup \{z,y\}$. The multiplicity of xz in $H(a,b,c,d)$ is two, so there are four missing edges in the subgraph of H induced by $S \cup \{x,z\}$: xab, zab and two edges containing x and z and one vertex in S. There are three non-isomorphic ways for this to occur: xzc, xzd or xza, xzc or xza, xzb. For the first, the four missing edges would be $\{xab, zab, xzc, xzd\}$ in which case there would be a copy of $F(3,3)$ with special edge xzb. The same thing would occur for the second possibility. So the four missing edges must be $\{xza, xzb, xab, zab\}$ and this must be true for each $x, z \in A$. Similarly, for each $u, v \in B$, the four missing edges in the subgraph of H induced by $S \cup \{u,v\}$ are $\{uvc, uvd, ucd, vcd\}$. By the inductive hypothesis, we know that $H \setminus S$ is isomorphic to $S(n-4)$. To complete the proof, we need to show this $S(n-4)$-subgraph "fits together" with the edges of H which intersect S to form $S(n)$, that is we need to show each edge of H disjoint from S intersects both A and B (since we already know each edge of H which hits S intersects both $A \cup \{a,b\}$ and $B \cup \{c,d\}$). If $x, y, z \in A$, then $xac, xad, xcd, yac, yad, ycd, zac, zad, zcd$ are all edges of H, so if xyz is also an edge we have an $F(3,3)$-subgraph, with an identical argument if $x, y, z \in B$. ■

4 Final Comments and Further Problems

$F(3,3)$ is critically 3-colorable, meaning it is not 2-colorable but deleting any edge results in a 2-colorable 3-graph. Let H be any critically 3-colorable 3-graph. Since $S(n)$ is 2-colorable it is certainly H-free, so $\pi(H) \geq \frac{3}{4}$. It may well be true that $\pi(H)$ must be equal to $\frac{3}{4}$. Sidorenko [15] showed that the stronger statement that $\mathrm{ex}(n;H) = s(n)$ for sufficiently large n is not necessarily true; his example is $H = K_5^{(3)}$ and n odd.

Let $\mathcal{H}$ be the family of all 3-graphs H on $[6]$ which have 10 edges, precisely one edge from each complementary pair of sets in $[6]^{(3)}$. Of course $F(3,3)$ is in $\mathcal{H}$. It would be interesting to determine $\pi(H)$ for other 3-graphs $H \in \mathcal{H}$. One is $H(6) = \{123, 126, 135, 234, 145, 146, 245, 256, 346, 356\}$. An equiblowup of $H(6)$, with the construction repeated in each of the parts, is the conjectured 3-graph with the maximum number of edges and no copy of $\{123, 124, 134\}$ (see [5]). Like $F(3,3)$, it is self-complementary in $[6]$ and critically 3-colorable.

The 3-graph $\{123, 124, 125, 126, 134, 135, 136, 234, 235, 236\}$ is another self-complementary member of $\mathcal{H}$, but it is 2-colorable (the partition $\{1,2\}$, $\{3,4,5,6\}$). Another interesting member of $\mathcal{H}$ is the star $S_6 = \{123, 124, 134, 125, 135, 145, 126, 136, 146, 156\}$. As pointed out in [10], if P is the set of all triples of noncollinear points in a projective plane of order 3, then the link-graph of each vertex is K_5-free, so taking an equiblowup of P, with the construction repeated within each part, then an equiblowup again, and so on, yields a sequence of S_6-free 3-graphs with limiting density $\frac{9}{14}$. Hence $\pi(S_6) \geq \frac{9}{14}$, and it is conjectured that equality holds.

Acknowledgements

We thank both referees for their detailed comments which could only come from an exceptionally careful reading of the manuscript.

References

[1] B. Bollobas. *Combinatorics*. Cambridge University Press, 1986.

[2] J. A. Bondy and Z. Tuza. A weighted generalization of Turán's theorem. *Journal of Graph Theory*, pages 267–275, 1997.

[3] D. de Caen and Z. Füredi. The maximum size of 3-uniform hypergraphs not containing a Fano plane. *Journal of Combinatorial Theory Series B*, pages 274–276, 2000.

[4] R Diestel. *Graph Theory, 2^{nd} edition*. Springer, 1991.

[5] P. Frankl and Z. Füredi. An exact result for 3-graphs. *Discrete Math*, 50:323–328, 1984.

[6] P. Frankl and Z. Füredi. A new generalization of the Erdos-Ko-Rado theorem. *Combinatorica*, 3:341–349, 2003.

[7] Z. Füredi and A. Kundgen. Turán problems for weighted graphs. *Journal of Graph Theory*, 40:195–225, 2002.

[8] Z. Füredi and M. Simonovits. Triple systems not containing a Fano configuration. *Combinatorics, Probability and Computing*, 14:467–484, 2005.

[9] J. Goldwasser. On the Turán Number of {123, 124, 345}. preprint.

[10] J. Goldwasser. Communication at AIM Workshop on Turan Hypergraphs. Palo Alto, 2011.

[11] P. Keevash and D. Mubayi. Stability results for cancellative hypergraphs. *Journal of Combinatorial Theory Series B*, 92:163–175, 2004.

[12] P. Keevash and D. Mubayi. The Turán number of $F_{3,3}$. 21:451–456, 2012.

[13] P. Keevash and B. Sudakov. The exact Turán number of triple systems. *Combinatorica*, 25:561–574, 2005.

[14] D. Mubayi and V. Rödl. On the turán number of triple systems. *Journal of Combinatorial Theory Series A*, 100:136–152, 2002.

[15] A. Sidorenko. What we know and what we do not know about Turán numbers. *Graphs and Combinatorics*, 11:179–199, 1995.