Optimization of Cubic Polynomial Functions without Calculus

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In every beginning calculus class, students learn to find extreme values of functions. This process helps students understand the essence of the derivative and is straightforward, especially when the functions are polynomials. In algebra and precalculus courses, students are often asked to find extreme values of polynomial functions in the context of solving an applied problem; but without the notion of derivative, something is lost. Either the functions are reduced to quadratics, since students know the formula for the vertex of a parabola, or solutions are approximated using a graphing calculator. In this article, we show that it is possible to find the relative maximum and minimum values of a cubic polynomial without appealing to the derivative or using a calculator to find an approximation. Instead, we will use elementary techniques that are found in, and are appropriate for, any advanced algebra or precalculus course. We start by solving the problem with calculus to give us the form of the solution.

THE CALCULUS SOLUTION

To find the extreme values of a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$, we can take the derivative to determine where the graph has a relative maximum or minimum. The derivative of this function is given by

$$f'(x) = 3ax^2 + 2bx + c,$$

and we see that the zeros of this function tell us the possible relative extreme values of the graph of f. Our old friend the quadratic formula tells us that the derivative of f will be zero (the only possible locations for extrema of f, since it is a polynomial) whenever

$$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}. \qquad (1)$$

Given that these are two distinct real numbers, we can use the second derivative test or just check the function values to determine which value gives a relative maximum and which value gives a relative minimum. This solution has the benefit of being concise, but it requires techniques beyond the scope of an algebra or precalculus course. The noncalculus solution, on the other hand, demonstrates the use of a variety of common algebra concepts that are taught in courses before calculus but are seldom applied to solving mathematical problems.

THE NONCALCULUS SOLUTION

In order to solve this problem without the use of calculus, we use some basic graph transformations, ideas about roots of polynomials, and the definition of an odd function. The first step is to show that every cubic polynomial graph has rotational symmetry about a point. We then shift our original function twice to create a new function. Finally, we see that one of the roots of this new function corresponds to the x-coordinate of a maximum or minimum value of our original function. In the process, we recover equation (1).

Step 1: Rotational Symmetry

We start by using a pair of function transformations to show that the graph of a cubic polynomial has rotational symmetry about a point. This first step allows us to begin examining the behavior of a function by considering a related function that is more obviously well behaved.

Recall that a function f is odd whenever $f(x) = -f(-x)$ for all x in the domain of f. Notice that $-f(-x)$ can be thought of as a combination of two transformations of the graph of f, a reflection across the y-axis followed by a reflection across the x-axis. From a geometric viewpoint, the composition of these two reflections amounts to a 180° rotation about the origin.

Using calculus techniques, de Villiers (2004) showed that every cubic polynomial of the form $f(x) = ax^3 + bx^2 + cx + d$ has rotational symmetry about a point on the line

$$x = \frac{-b}{3a}.$$

Our solution to the extrema problem starts by providing a noncalculus derivation of de Villiers's result. First, note that a cubic function $f(x) = ax^3 + bx^2 + cx + d$ is odd if and only if

$$\begin{aligned} ax^3 + bx^2 + cx + d &= -[a(-x)^3 + b(-x)^2 + c(-x) + d] \\ &= ax^3 - bx^2 + cx - d, \end{aligned}$$

which is true if and only if $b = d = 0$.

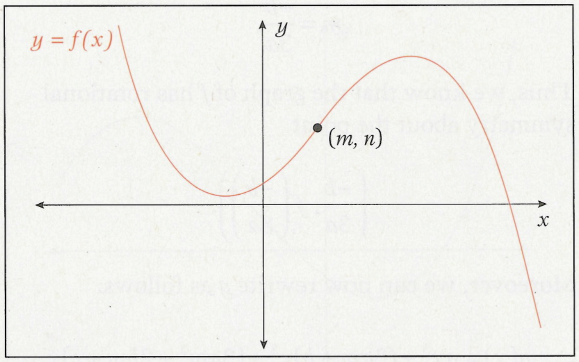

Fig. 1 The graph of a cubic polynomial function f

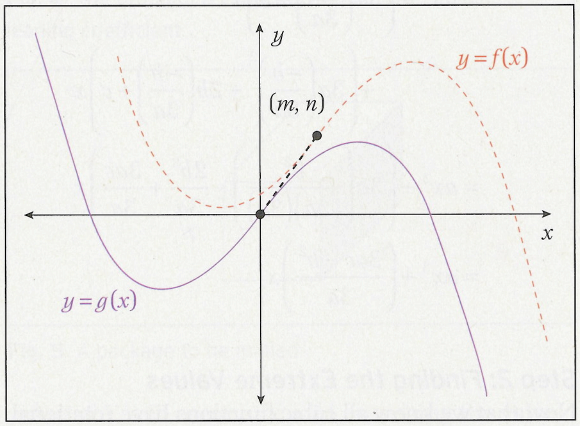

Fig. 2 The graph of g, given by $g(x) = f(x + m) - n$

More generally, we can show that every cubic polynomial has rotational symmetry about a point by first noticing that if $f(x) = ax^3 + bx^2 + cx + d$ is a cubic polynomial, then the graph of f will have rotational symmetry about the point (m, n) if and only if the function g given by $g(x) = f(x + m) - n$ is an odd function. Notice that this pair of transformations moves the point (m, n) on the graph of f to the origin. We show a sample graph of a cubic in **figure 1** and its transformation in **figure 2**.

By our earlier observation, we know that g will be odd if and only if g has no constant term and the coefficient of x^2 is zero. Now, since $n = f(m)$, we have

$$n = am^3 + bm^2 + cm + d.$$

So we can rewrite g as follows:

$$\begin{aligned} g(x) &= f(x+m) - n \\ &= a(x+m)^3 + b(x+m)^2 + c(x+m) + d \\ &\quad - (am^3 + bm^2 + cm + d) \\ &= a(x^3 + 3x^2m + 3xm^2 + m^3) + b(x^2 + 2xm + m^2) \\ &\quad + c(x+m) + d - am^3 - bm^2 - cm - d \\ &= ax^3 + 3ax^2m + 3axm^2 + bx^2 + 2bxm + cx \\ &= ax^3 + (3am + b)x^2 + (3am^2 + 2bm + c)x \end{aligned}$$

But we know that g is odd if and only if the coefficient of x^2 is zero. This means we must have

$$m = \frac{-b}{3a}.$$

Thus, we know that the graph of f has rotational symmetry about the point

$$\left(\frac{-b}{3a}, f\left(\frac{-b}{3a}\right)\right).$$

Moreover, we can now rewrite g as follows:

$$\begin{aligned} g(x) &= ax^3 + (3am + b)x^2 + (3am^2 + 2bm + c)x \\ &= ax^3 + \left(3a\left(\frac{-b}{3a}\right) + b\right)x^2 \\ &\quad + \left(3a\left(\frac{-b}{3a}\right)^2 + 2b\left(\frac{-b}{3a}\right) + c\right)x \\ &= ax^3 + \left(3a\left(\frac{b^2}{(3a)(3a)}\right) - \frac{2b^2}{3a} + \frac{3ac}{3a}\right)x \\ &= ax^3 + \left(\frac{3ac - b^2}{3a}\right)x \end{aligned}$$

Step 2: Finding the Extreme Values

Now that we know all cubic functions have rotational symmetry, we can use what we know about roots of polynomials to find the relative maximum and minimum values of a cubic polynomial. In this step, we build upon our previous observations and transform the function g into a new function h whose factorization we can determine using the factor theorem.

Recall that if $f(x) = ax^3 + bx^2 + cx + d$ with

$$m = \frac{-b}{3a}$$

and $n = f(m)$, then $g(x) = f(x + m) - n$ is an odd function and so $g(0) = 0$. If g is to have a relative maximum (and a relative minimum), then g must have two additional real roots that are symmetric about the origin. Suppose that g has a relative maximum at (q, M) and we define h by

$$h(x) = g(x) - M.$$

Since g has a relative maximum at $x = q$, we know that the graph of h will look like the one shown in **figure 3**. Moreover, we know that h has a double root at $x = q$ and one other real root, say $x = p$. Thus, we can write $h(x) = a(x - q)^2(x - p)$. That is,

$$h(x) = ax^3 + (-2aq - ap)x^2 + (aq^2 + 2apq)x - apq^2.$$

But $h(x) = g(x) - M$, and

$$g(x) - M = ax^3 + \left(\frac{3ac - b^2}{3a}\right)x - M.$$

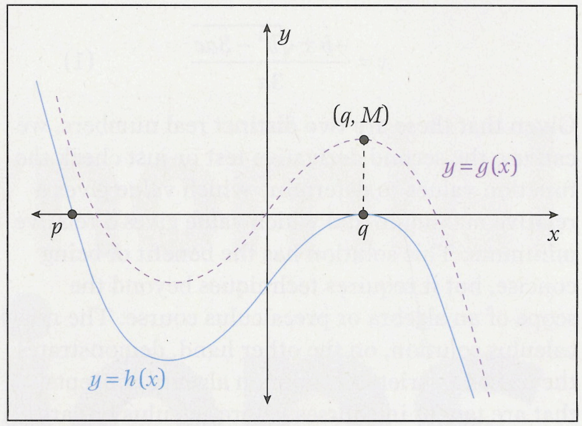

Fig. 3 The graph of h, given by $h(x) = g(x) - M$

Equating coefficients of x^2 and x gives us the following two equations:

$$-2aq - ap = 0 \qquad (2)$$

$$aq^2 + 2apq = \frac{3ac - b^2}{3a} \qquad (3)$$

Equation (2) gives us $p = -2q$. Substituting this into equation (3), we get

$$aq^2 + 2a(-2q)q = \frac{3ac - b^2}{3a}.$$

Solving this equation for q gives us

$$q = \pm\frac{\sqrt{b^2 - 3ac}}{3a}.$$

Thus, the relative maximum value of g is at the point (q, M) for the appropriate choice of q, given that $b^2 - 3ac > 0$. In the case where $b^2 - 3ac = 0$, we see that $q = 0$; and so our graph would be forced to have a relative maximum at the origin, which is impossible because g is odd. Thus, our function f would have no relative extrema. If $b^2 - 3ac < 0$, then q would be undefined, and so the graph of f would again have no maximum or minimum values. Since our original assumption was that we had a function with relative extrema, we know that the value of $b^2 - 3ac$ must be positive. Moreover, because of the symmetry of the graph of g, we know that one value of q will give us the x-coordinate of the relative maximum of g and the other value of q will give us the x-coordinate of the relative minimum of g. To determine which value of q yields the relative maximum and which the relative minimum, we need only consider the value of a. If $a < 0$, then we know that the long-run behavior of f (and also g) will be that

$$\lim_{x \to \infty} f(x) = -\infty.$$

Thus, the corresponding graph would look similar to the graph shown in **figure 1**. On the other hand, if $a > 0$, then the long-run behavior of f (and g) would be

$$\lim_{x \to \infty} f(x) = \infty$$

and so the graph of f would be similar to the graph shown in **figure 4**. Therefore, if $a > 0$, then the positive value of q would yield the x-coordinate of the relative maximum of g, and the negative value of q would yield the x-coordinate of the relative minimum of g. If $a < 0$, then the roles of the positive and negative values of q would be reversed.

If we had chosen to look for a relative minimum, we would have shifted the graph of h upward instead of downward. The function h would still have a double root at $x = q$ and one other root, so the calculations above would follow through without change.

Putting It All Together

Now, since we were originally concerned with finding the relative maximum value of f, we need to translate the x-value of the relative maximum of g back to correspond to the graph of f. That is, since the graph of f has its point of rotational symmetry on the line $x = m$, the graph of f will have a relative maximum at the point $(m + q, f(m + q))$. But notice that since $x = m + q$, this gives us

$$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}.$$

This is equation (1).

AN EXAMPLE

One of the standard optimization problems students encounter concerns a package that is to be sent through the mail (see **fig. 5**). Some postal guidelines require that the combined length and girth of a package with a square cross-section cannot exceed a fixed amount—for example, 84 inches. (The girth is the perimeter of the square end.) The question, of course, is what is the largest package (in terms of volume) that you can send? Using a bit of analytic geometry, we see that we can answer this question by finding the maximum value of $V(x) = 84x^2 - 4x^3$, where x is the length of the square end of the box. The derivative of V is given by $V'(x) = 168x - 12x^2$ and will be zero whenever $x = 0$ or $x = 14$. Since the coefficient of x^3 is negative, we know that the larger value, $x = 14$, gives the relative maximum.

Using the techniques derived in the previous sections of this article, we see that the function V will have a relative maximum at the point whose x-value is given by

$$x = \frac{-84 \pm \sqrt{84^2 - 3(-4)(0)}}{3(-4)}.$$

Again, since the leading coefficient is negative, we know that the relative maximum will be at the larger x-value. Therefore, the relative maximum

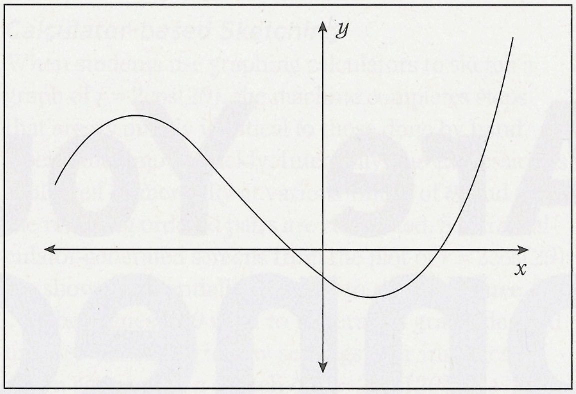

Fig. 4 The graph of a cubic polynomial with a positive leading coefficient

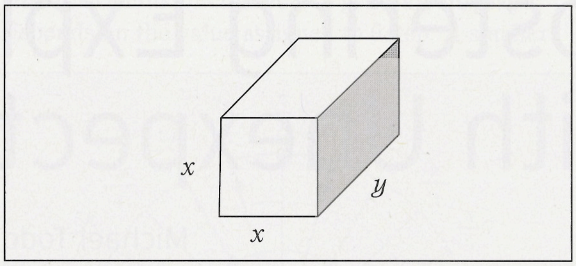

Fig. 5 A package to be mailed

value of V will occur at $x = 14$, matching the value given by the calculus solution.

CONCLUSION

We have seen that using some basic precalculus techniques, we can find the relative maximum and minimum of a cubic polynomial function. In this we have found a solution to a problem that previously required the use of calculus. This new method is applicable to any precalculus course and demonstrates the usefulness of a variety of algebraic techniques in new and interesting ways.

REFERENCES

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