Section 5.5 The Substitution Rule

It's easy enough to calculate integrals when we know particular antiderivatives:

For Example
$$\int cosx\,dx = \sin x + c$$
OR
$$\int x^2 dx = \frac{x^3}{x} + c$$

But, what do we do when the antiderivative is not obvious?

For Example $\int 3x^2\sqrt{1+x^3}\,dx$

One way is to try to find a substitute for the integrand.

Let $u = 1+x^3$

Note: $\frac{du}{dx} = 3x^2$

$du = 3x^2 dx$

Notice then, that:

$$\int 3x^2\sqrt{1+x^3}\,dx = \int \sqrt{1+x^3}\cdot 3x^2\,dx$$

$= \int \sqrt{u}\,du$ ← and this integral we can do

$$= \int u^{\frac{1}{2}} du$$

$$= \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}\left(1+x^3\right)^{\frac{3}{2}} + c$$

The idea is that you are doing the chain rule backwards:

You already know the Chain Rule: $\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right)g'(x)$

Notice:
$$\int f'\left(g(x)\right)g'(x)\,dx = \int \frac{d}{dx}\left[f\left(g(x)\right)\right]dx$$
$$= f\left(g(x)\right)$$

So, if I go back to my earlier example:

$$\int 3x^2\sqrt{1+x^3}\,dx = \frac{2}{3}\left(1+x^3\right)^{\frac{3}{2}} + c$$

I should be able to regenerate through the derivative, Let's check:

$$\frac{d}{dx}\left(\frac{2}{3}(1+x^3)^{\frac{3}{2}}+c\right)$$

$$=\frac{2}{3}\left(\frac{3}{2}\right)(1+x^3)^{\frac{1}{2}}(3x^2)+0$$

$$=(1+x^3)^{\frac{1}{2}}(3x^2)\qquad \text{tada!}$$

The hard part to the method is training yourself to identify the "u" to start with. Any ideas?
Note that $u=g(x)$

Key: The *u* is the "inside" function.

Let's do some examples.

Pg. 298 #6) $\int e^{\sin x}\cos x\,dx=?$

Let $u=\sin x$
$du=\cos x\,dx$
$=\int e^u\,du=e^u+c=e^{\sin x}+c$

Pg. 298 #12) $\int (2-x)^6\,dx=?$

Let $u=2-x$
$du=-1\,dx$ ← this time we do not have a -1 already

So we have to force it in without changing value

$$\int (2-x)^6\,dx=-\int (2-x)^6(-1)\,dx$$
$$=-\int u^{\text{¿}}\,du$$
$$=-\frac{u^7}{7}+c$$
$$=-\frac{(2-x)^7}{7}+c$$

Pg. 299 #14) $\int \frac{x}{(x^2+1)^2}\,dx=?$

Let $u=x^2+1$
$du=2x\,dx$

$$\int \frac{x}{(x^2+1)^2}dx = \frac{1}{2}\int \frac{1}{(x^2+1)^2} \cdot 2x\,dx$$

$$\frac{1}{2}\int \frac{1}{u^2} \cdot du$$

$$\frac{1}{2}\int u^{-2}du$$

$$= \frac{1}{2}\frac{u^{-1}}{-1}+c = \frac{1}{2u} = -\frac{1}{2(x^2+1)}+c$$

$$\int \tan x\,dx = ?$$

Now, this one is different, isn't it? The inside function is x and that's not going to get me anywhere! Ideas? Let's break into trig components...

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$$

We still do not have “obvious” inside / outside functional relationships, but what do we know about $\sin x$ and $\cos x$ in terms of their derivatives?

$$u = \cos x \qquad \text{or} \qquad u = \sin x$$
$$du = -\sin x\,dx \qquad\qquad du = \cos x\,dx$$

Which one of these will help us? $u = \cos x$

Why? $\frac{\sin x\,dx}{\cos x}$ versus $\sin x \cdot \frac{1}{\cos x}dx$ *du* has been broken up!

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = \int \frac{1}{\cos x} \cdot \sin x\,dx$$

So: $$= -\int \frac{1}{\cos x} \cdot (-\sin x)\,dx$$

$$= -\int \frac{1}{u}du = -\ln|u|+c = -\ln|\cos x|+c$$

OK, now a reminder:

$-\ln|u| = \ln|u|^{-1}$ because of log rules (look at pg. 155)

$$-\ln|\cos x| = \ln|\cos x|^{-1}$$

So, $$= \ln\left|\frac{1}{\cos x}\right|$$

$$= \ln|\sec x|$$

That means, depending on what book you use, you may see:

$$\int \tan x\,dx = -\ln|\cos x|+c = \ln|\sec x|+c$$

So far we have only use the substitution rule on indefinite integrals. How will things change when we use the definite forms (i.e. stuff in limits now).

By example:

#38) $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = ?$

Let $u = x^2$
$du = 2x\,dx$

$$\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \frac{1}{2}\int_0^{\sqrt{\pi}} \cos(x^2)\cdot 2x\,dx$$
$$= \frac{1}{2}\int_{\text{What goes here?}}^{\text{What goes here?}} \cos(u)\,du$$

Two Alternatives:

One Way: You must show x = (or whatever variable is at play) when you work this way.

$$= \frac{1}{2}\int_{x=0}^{x=\sqrt{\pi}} \cos u\,du$$
$$= \frac{1}{2}\sin u\Big|_{x=0}^{x=\sqrt{\pi}}$$
$$= \frac{1}{2}\sin(x^2)\Big|_0^{\sqrt{\pi}}$$
$$= \frac{1}{2}\left(\sin(\sqrt{\pi}^2) - \sin(0^2)\right)$$
$$= \frac{1}{2}\left(\sin(\pi) - \sin(0)\right) = \frac{1}{2}(0-0) = 0$$

Another Way: A true change of variable...

From: $u = x^2$ $\quad x = 0 \Rightarrow u = (0)^2$
$du = 2x\,dx$ $\quad x = \sqrt{\pi} \Rightarrow u = \left(\sqrt{\pi}\right)^2 = \pi$

$$= \frac{1}{2}\int_0^{\pi} \cos(u)\,du$$
$$= \frac{1}{2}\sin u\Big|_0^{\pi}$$
$$= \frac{1}{2}\left(\sin(\pi) - \sin(0)\right)$$
$$= \frac{1}{2}(0-0)$$
$$= 0$$

pg. 299 #42) $\int_0^{\frac{\pi}{2}} \cos x \sin(\sin x)\,dx$

Let $u = \sin x$

$du = \cos x\, dx$

$x = 0 : u(0) = \sin(0) = 0$

$x = \frac{\pi}{2} : u\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$$\int_0^{\frac{\pi}{2}} \cos x \sin(\sin x)\,dx = \int_0^{\frac{\pi}{2}} \sin(\sin x)\cos x\,dx$$

$$= \int_0^1 \sin(u)\,du$$

$$= -\cos(u)\Big|_0^1$$

$$= -(\cos(1) - \cos(0))$$

$$= \mathbf{1 - \cos(1)}$$

#48) $\int_0^a x\sqrt{a^2 - x^2}\,dx$ where a is a constant

Let $u = a^2 - x^2$

$du = -2x\,dx$

$x = 0 : u(0) = a^2 - 0^2 = a^2$

$x = a : u(a) = a^2 - a^2 = 0$

$$\int_0^a x\sqrt{a^2 - x^2}\,dx = -\frac{1}{2}\int_0^a \sqrt{a^2 - x^2}(-2x)\,dx$$

$$-\frac{1}{2}\int_{a^2}^0 \sqrt{u}\,du = -\frac{1}{2}\int_{a^2}^0 u^{\frac{1}{2}}\,du$$

$$= +\frac{1}{2}\int_0^{a^2} u^{\frac{1}{2}}\,du \quad \text{Flipped Limits}$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{\frac{3}{2}}\Big|_0^{a^2}$$

$$= \frac{1}{3}\left((a^2)^{\frac{3}{2}} - (0)^{\frac{3}{2}}\right)$$

$$= \frac{1}{3}(a^3 - 0)$$

$$= \frac{a^3}{3}$$

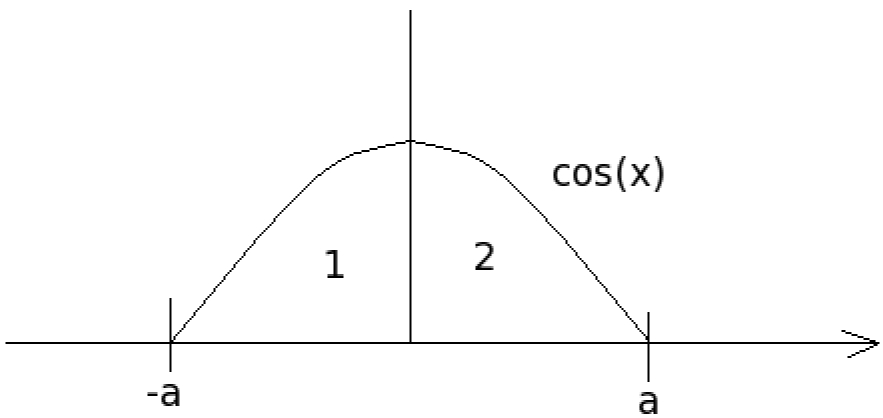
Handy Info:
Function symmetry can make integral evaluation more easy (but I only use this when it's deadly obvious)

1. If *f* is continuous on $[-a\ ,\ a]$ and *f* is even, then:

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Why? What does this even mean? $f(x) = f(-x)$
Symmetry over the y-axis (like cosine)

Picture:



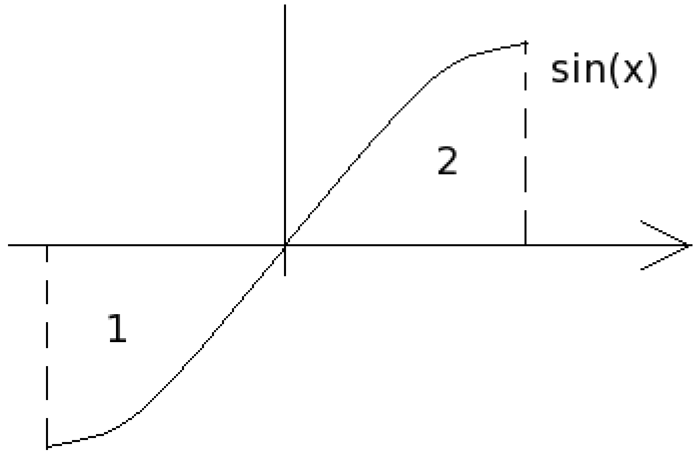
Area 1 = Area 2, hence we can double one side for total area

2. If *f* is continuous on $[-a\ ,\ a]$ and *f* is odd, then:

$$\int_{-a}^{a} f(x)\,dx = 0$$

Why? What does odd mean? $f(x) = -f(x)$
Symmetry over the origin (like sin x)

Picture:



Area 1 = - Area 2, hence added together the total area = 0