Section 5.3 Evaluation of Integrals

Theorem If *f* is continuous on the interval $[a\ ,\ b]$, then: $\int_a^b f(x)\,dx = F(b) - F(a)$

where *F* is any antiderivative of *f*, that is $F' = f$

So, what this means is that an integral can be evaluated by subtracting the endpoint values of *F*, the antiderivative of *f*.

Example What is the antiderivative of: $f(x) = x^3 + x$?

$$F(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + c$$

Now chose $c = 0$; what is $\int f(x)\,dx$ on $[2\ ,\ 4]$?

$$\int_2^4 \left(x^3 + x^2\right) dx = \left.\frac{1}{4}x^4 + \frac{1}{3}x^3\right]_2^4 \leftarrow \text{symbol means evaluated at F(4) - F(2)}$$

So:

$$= \frac{1}{4}(4)^4 + \frac{1}{3}(4)^3 - \left[\frac{1}{4}(2)^4 + \frac{1}{3}(2)^3\right]$$

$$= 64 + \frac{64}{3} - \left(4 + \frac{8}{3}\right)$$

$$= 60 + \frac{56}{3} = \frac{180 + 56}{3} = \frac{\mathbf{236}}{3}$$

Example Evaluate $\int_1^3 e^x\,dx$

$$\int_1^3 e^x\,dx = \left.e^x\right]_1^3 = e^3 - e^1$$

Remember the antiderivative of $e^x = e^x$!

The Big Concept:

Book Notation: If $F'(x) = f(x) \Rightarrow \int f(x)\,dx = F(x)$

Differentiation and integration are linked in this way.

Notice that by not having limits on our integral it is described differently – as <u>indefinite.</u>

<u>Indefinite</u> integrals are <u>Functions.</u>

<u>Definite</u> integrals are <u>Numbers.</u>

From Table pg. 277 (You need to know this!)

1. $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

2. $\int c\,f(x)\,dx = c\int f(x)\,dx$

3. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1}+c \qquad (n\neq 1)$

4. $\int x^{-1}\,dx = \int \dfrac{1}{x}\,dx = \ln|x|+c$

5. $\int e^x\,dx = e^x+c$

6. $\int a^x\,dx = \dfrac{a^x}{\ln a}+c$

7. $\int \sin x\,dx = -\cos x+c$

8. $\int \cos x\,dx = \sin x+c$

9. $\int \sec^2 x\,dx = \tan x+c$

10. $\int \csc^2 x\,dx = -\cot x+c$

11. $\int \sec(x)\cdot\tan(x)\,dx = \sec x+c$

12. $\int \csc x\cot x\,dx = -\csc x+c$

13. $\int \dfrac{1}{x^2+1}\,dx = \tan^{-1}x+c = \arctan x+c$ Note: $\tan^{-1}x \neq \dfrac{1}{\tan x}$

14. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$

Notice that everyone of these indefinite forms carries an unknown constant. This is only true for integrals without limits. Once the limits are known, and the integrals are definite we no longer need *c*. Remember, in that situation the integrals represent particular values.

Example $\int x\sqrt{x}\,dx = \int x \cdot x^{\frac{1}{2}}\,dx = \int x^{\frac{3}{2}}\,dx = \frac{1}{\frac{5}{2}} x^{\frac{5}{2}} + c = \frac{2}{5} x^{\frac{5}{2}} + c$

Example pg. 282 #4) $\int_{-2}^{0} \left(u^5 - u^3 + u^2\right) du = ?$

$$\int_{-2}^{0} \left(u^5 - u^3 + u^2\right) du = \int_{-2}^{0} u^5\,du - \int_{-2}^{0} u^3\,du + \int_{-2}^{0} u^2\,du$$

$$= \frac{1}{6}u^6\Big|_{-2}^{0} - \frac{1}{4}u^4\Big|_{-2}^{0} + \frac{1}{3}u^3\Big|_{-2}^{0}$$

$$= \frac{1}{6}\left(0^6 - (-2)^6\right) - \frac{1}{4}\left(0^4 - (-2)^4\right) + \left(0^3 - (-2)^3\right)$$

$$= \frac{1}{6}(0-64) - \frac{1}{4}(0-16) + \frac{1}{3}(0-(-8))$$

$$= -\frac{32}{3} + 4 + \frac{8}{3}$$

$$= -\frac{24}{3} + \frac{12}{3} = \mathbf{-\frac{12}{3}}$$

Example pg. 282 #12) $\int_{1}^{2} \frac{y + 5y^7}{y^3}\,dy = ?$

$$\int_{1}^{2} \frac{y + 5y^7}{y^3}\,dy = \int_{1}^{2} \left(\frac{1}{y^2} + 5y^4\right) dy$$

$$= \int_{1}^{2} y^{-2}\,dy + 5\int_{1}^{2} y^4\,dy$$

$$\frac{1}{-1}y^{-1}\Big|_{1}^{2} + 5\left(\frac{1}{5}y^5\right)\Big|_{1}^{2}$$

$$= \frac{-1}{y}\Big|_{1}^{2} + y^5\Big|_{1}^{2}$$

$$= \left(\frac{-1}{2} - \left(\frac{-1}{1}\right)\right) + \left(2^5 - 1^5\right)$$

$$= \frac{1}{2} + 32 - 1 = 32 - \frac{1}{2}$$

$$= \frac{64}{2} - \frac{1}{2} = \mathbf{\frac{63}{2}}$$

Example #20) $\int_0^1 \frac{4}{t^2+1}\, dt = ?$

$$\int_0^1 \frac{4}{t^2+1}\, dt = 4\int_0^1 \frac{1}{t^2+1}\, dt$$

$$= 4\left[\tan^{-1}(x)\right]_0^1$$

Tricky part, domain of $\tan^{-1}(x) = \left(\frac{-\pi}{2}\ ,\ \frac{\pi}{2}\right)$

$$= 4\left(\frac{\pi}{4} - 0\right)$$

$$= \pi$$

Example #26) $\int_0^{\frac{\pi}{3}} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\, d\theta = ?$

$$\int_0^{\frac{\pi}{3}} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\, d\theta = \int_0^{\frac{\pi}{3}} \frac{\sin\theta\left(1+\tan^2\theta\right)}{\sec^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin\theta\left(sec^2\theta\right)}{\sec^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \sin\theta\, d\theta$$

$$= -\cos\theta\Big|_0^{\frac{\pi}{3}}$$

$$= -\left(\cos\left(\frac{\pi}{3}\right) - \cos(0)\right)$$

$$= -\left(\frac{1}{2} - 1\right)$$

$$= -\left(-\frac{1}{2}\right)$$

$$= \frac{1}{2}$$

Example #48) The current in a wire is defined as the derivative of the charge: $I(t) = Q'(t)$.

What does $\int_a^b I(t)\,dt$ represent?

Remember the big concept: $F'(x) = f(x) \Rightarrow \int f(x)\,dx = F(x)$

So: for this problem.

$$Q'(t) = I(t) \Rightarrow \int I(t)\,dt = Q(x)$$

$$\int_a^b I(t)\,dt = Q(b) - Q(a)$$

In words, the difference in the total charge between the points *a* and *b*.