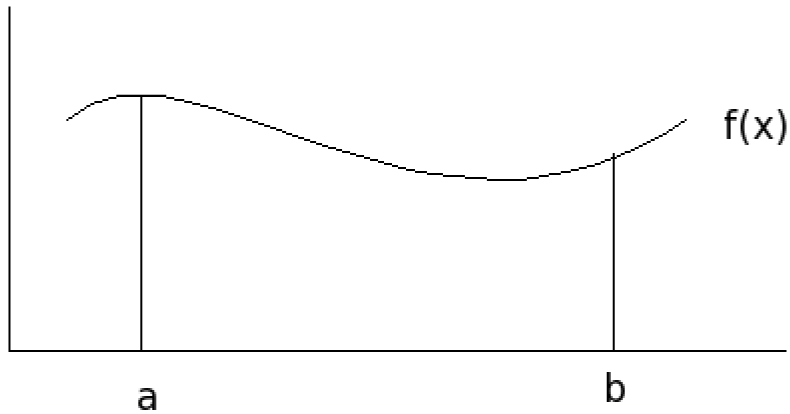
Section 5.4 The Fundamental Theorem of Calculus

A Brief Review of Sections 5.1 through 5.3



Theorem If f is integrable on $[a\ ,\ b]$, then

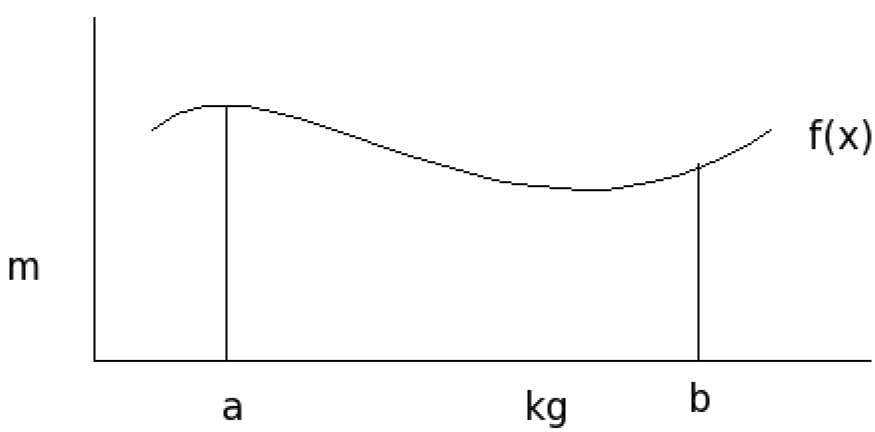
$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a+i\,\Delta x$

Recall that we were adding up the areas of a whole bunch $(n\to\infty)$ of small rectangles to calculate the integral (idea Riemann Sum) LH rule, RH rule, MP rule.

So, integrals, in 2D, represent some kind of area. How do we know the units of that area?

For Example:



unit of area = *m-kg*

The next big concept was related to how we evaluate a definite integral (that means one with limits) without using rectangles.

Evaluation Theorem: If *f* is continuous on the interval $[a\ ,\ b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where *F* is the antiderivative of *f*. $(F' = f)$

*Essentially then we started to do derivatives backwards

Along the way, we learned a whole bunch of integral properties as well...you need to remember these.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$ flipping limits flips sign

$$\int_a^a f(x)dx = 0$$ There is no area under the curve at a single point

$$\int_a^b f(x)dx \ + \ \int_b^c f(x)dx = \int_a^c f(x)dx$$ Extension of limits

$$\int_a^b c\,f(x)dx = c\int_a^b f(x)dx$$ Constants can come out multiplicatively

$$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \ \pm \ \int_a^b g(x)dx$$ adding and subtracting

One other big thing, you absolutely need to know he indefinite integral forms (pg. 277) as these are the building blocks of more and more difficult integrals we will soon begin to build.

$$\int x^n = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\frac{\int 1}{x} dx = \ln|x| + c$$

$$\int e^x dx = e^x + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

$$\int \sin x\, dx = -\cos x + c \qquad \int \cos x\, dx = \sin x + c$$
$$\int \sec^2 x\, dx = \tan x + c \qquad \int \csc^2 x\, dx = -\cot x + c$$
$$\int \sec x \tan x\, dx = \sec x + c \qquad \int \csc x \cot x\, dx = -\csc x + c$$
$$\int \frac{1}{x^2+1} dx = \tan^{-1} x + c \qquad \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$$

OK, onward and upward!

Section 5.4

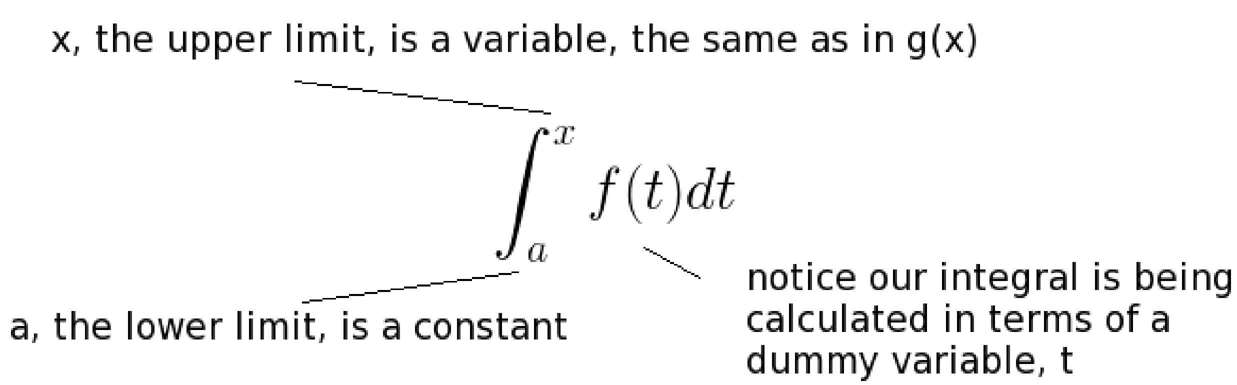
We start with the Theorem:

pg. 288 Suppose *f* is continuous on $[a\ ,\ b]$

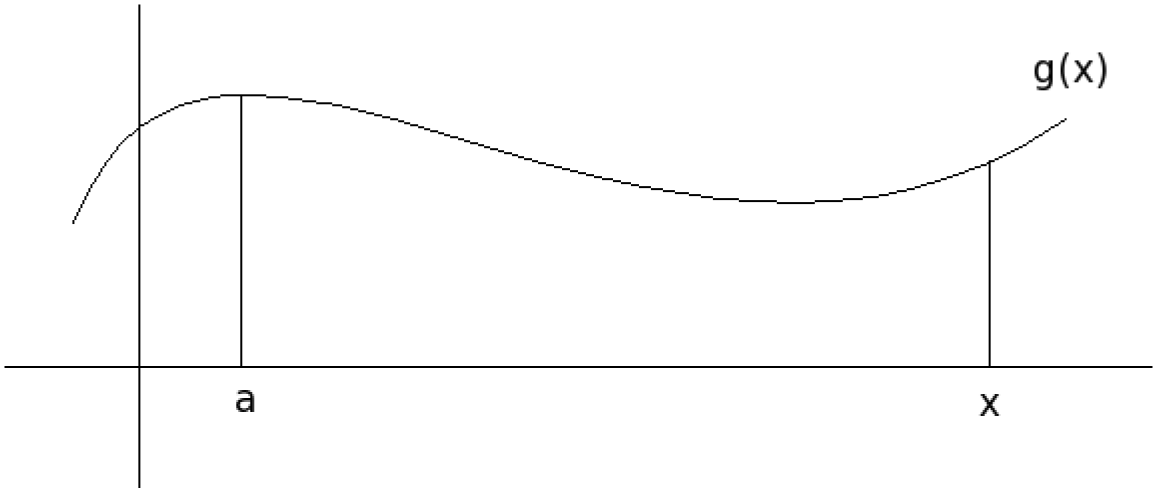
1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$
2. $\int_a^b f(x)dx = F(b) - F(a)$, where *F* is any anti-derivative of *f* $(F' = f)$

OK, so part (2) we've already figured out before as “the Evaluation Theorem” - so what does this pat (1) mean?

Let's look at it:



Visually:



The upper limit can be anything we assign

The book calls it “the area so far...”

like $x = b$
or $x = 7$
or $x = 1{,}000{,}000$

By using x we can calculate our integral very generally.

Example pg. 291 #4) $g(x) = \int_a^x (1 + \sqrt{t})\,dt$

$$= \int_1^x 1\,dt \;+\; \int_1^x t^{\frac{1}{2}}\,dt$$

$$= t + \left.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^x$$

$$= t + \left.\frac{2}{3}t^{\frac{3}{2}}\right|_1^x$$

$$= x + \frac{2}{3}x^{\frac{3}{2}} - \left(1 + \frac{2}{3}(1)^{\frac{3}{2}}\right)$$

$$g(x) = x + \frac{2}{3}x^{\frac{3}{2}} - \frac{5}{3}$$

So, we can play with this notationally as well.

$$g(2) = \int_a^{x=2} \left(1+\sqrt{t}\right) dt = (2) + \frac{2}{3}(2)^{\frac{3}{2}} - \frac{5}{3}$$

$$g(10) = \int_a^{x=10} \left(1+\sqrt{t}\right) dt = (10) + \frac{2}{3}(10)^{\frac{3}{2}} - \frac{5}{3}$$

The result is not anything shocking at all. It's really just another way to write a definite integral, but there are other interesting properties to exploit.

Example (From Above)

$$g(x) = \int_1^x \left(1+t^{\frac{1}{2}}\right) dt = x + \frac{2}{3}x^{\frac{3}{2}} - \frac{5}{3}$$

Now, find $g'(x)$ (The derivative)

$$g'(x) = 1 + \frac{2}{3}\left(\frac{3}{2}\right)x^{\frac{1}{2}} + 0$$

$$g'(x) = 1+x^{\frac{1}{2}}$$

Where have you seen that <u>very</u> recently? That's right...as the <u>integrand</u> (stuff inside the integral sign) of our original $g(x)$, just in terms of x instead of t.

That's why, in part 1 of the theorem:

If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$

The derivative of g with respect to x is the integrand, f with respect to x.

Example pg. 291 #6) Find $g'(x)$ for:

$$g(x) = \int_1^x \ln t\, dt \quad \Rightarrow \quad g'(x) = \ln x$$

The property avoids a seriously messy calculation.

I know right now you have not seen many functions defined by integrals in your coursework, but they are coming. They are very common in the sciences, particularly physics-the sister of engineering.

Book talks about Fresnel function – optics

Other Forms of FTC, part 1.

Example Find $g'(x)$ for:

$$g(x) = \int_0^{x^2} \sqrt{1+r^3}\,dr$$ Notice that we have x^2, not x. What do we do?

Think of it like this:

$$g'(x) = \frac{d}{dx}\int_0^{x^2} \sqrt{1+r^3}\,dr$$ invoke the $\frac{d}{dx}$ notation

let $u = x^2$

$$g'(x) = \frac{d}{dx}\int_0^{u} \sqrt{1+r^3}\,dr$$ but now our variables do not match

Use chain rule: $$\underbrace{\frac{d}{dx}}_{\text{What I have}} = \underbrace{\frac{d}{du}}_{\text{What I want}} \cdot \underbrace{\frac{du}{dx}}_{\text{Correction Factor}}$$

$$g'(x) = \frac{d}{du}\left[\int_0^{u} \sqrt{1+r^3}\,dr\right]\frac{du}{dx}$$

$$= \sqrt{1+u^3} \cdot \frac{du}{dx}$$ $\leftarrow$ What's this?

$$u = x^3$$
$$\frac{du}{dx} = 2x$$

Sub back:

$$g'(x) = \sqrt{1+\left(x^2\right)^3} \cdot 2x = 2x\sqrt{1+x^6}$$

Example pg. 291 #12) Find y' for:

$$y = \int_{e^x}^{0} \sin^3 t \, dt$$ notice we are integrating from a function to a constant, so we flip.

$$y = -\int_{0}^{e^x} \sin^3 t \, dt \qquad \frac{d}{dx} = \frac{d}{du} \cdot \frac{du}{dx}$$

Now, we invoke the $\frac{d}{dx}$ notation an make substitution for u.

Let $u = e^x \qquad \frac{du}{dx} = e^x$

$$y' = \frac{d}{dx}\left[-\int_{0}^{u} \sin^3 t \, dt\right]\frac{du}{dx}$$

Chain rule the derivative notation:

$$y = \frac{d}{du}\left[-\int_{0}^{u} \sin^3 t \, dt\right]\frac{du}{dx}$$

$$y' = -\sin^3 u\left(\frac{du}{dx}\right)$$

Sub in:

$$y' = -\sin^3\left(e^x\right) \cdot \left(e^x\right) = -e^x \sin^3\left(e^x\right)$$

Averaging with Integrals

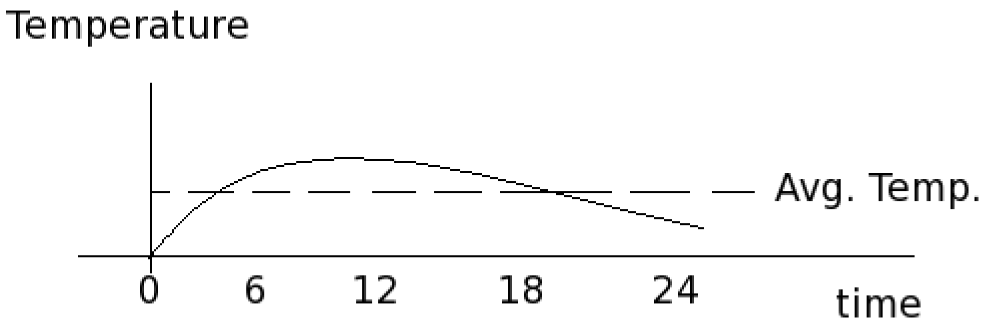
First, let's say you have 10 numbers, and you want their average. How do you find it?

You sum the numbers up and divide by 10.

So, if you have n numbers:

$$\overline{y} = \frac{y_1 + y_2 + y_3 + ... + y_n}{n}$$ $\overline{}$ means average

What happens, then, when n becomes infinite? This is the same idea as having continuous readings.



(Text pg. 289)

Let's call 0 = *a* and 24 = *b*

One way to start is to break the interval up into small pieces and take discrete readings at those points.

Relate: $\Delta x = \frac{b-a}{n} \quad \Rightarrow \quad n = \frac{b-a}{\Delta x}$

Approximating:

$$\underset{\text{approx}}{\overline{f(x)}} = \frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n}$$

So. $$= \frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{\frac{b-a}{\Delta x}}$$

$$= \frac{\Delta x}{b-a}\left[f(x_1)+f(x_2)+\ldots+f(x_n)\right]$$

Looking familiar yet? Distribute Δx through:

$$= \frac{1}{b-a}\left[f(x_1)\Delta x+f(x_2)\Delta x+\ldots+f(x_n)\Delta x\right]$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$

What happens as $n \to \infty$ (i.e. making exact)

$$\lim_{n\to\infty} \underset{\text{approx}}{\overline{f(x)}} = \frac{\lim_{n\to\infty} 1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$ Def. Of Integral

$$\overline{f(x)} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

So, that makes the average value of a continuous function, *f*, on an interval $[a\ ,\ b]$ is:

$$\overline{f(x)} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Mean Value Theorem for Integrals:

Furthermore, we can way:
If *f* is continuous on $[a\ ,\ b]$, then there exists a number *c* in $[a\ ,\ b]$ such that:

$$f(c) = \bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

more commonly written:

$$\int_a^b f(x)\,dx = (b-a)f(c)$$

Just means that an average value for a function exists on any closed interval and that the function holds that average value at at least one point.

Example Given $f(x) = \sqrt{x}$ on $[0\ ,\ 4]$

Find *c* such that $f(c) = \bar{f}$

1st Find, $\bar{f}$:

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{4-0}\int_0^4 x^{\frac{1}{2}}\,dx$$

$$= \frac{1}{4}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4$$

$$= \frac{1}{6}\left[4^{\frac{3}{2}} - 0\right] = \frac{1}{6}(8) = \frac{4}{3}$$

2nd Find *c*:

$$f(c) = \sqrt{c} = \frac{4}{3}$$

$$c = \frac{16}{9}$$