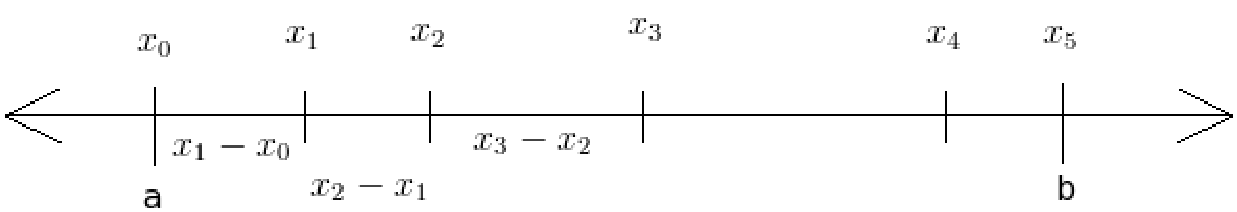
Section 5.2 The Definite Integral

As a reminder, in section 5.1 we talked about calculating the area under a curve by adding up the areas of little rectangles, equally spaced, that we made by cutting up an interval into *N* equal pieces. We determined that the area could be estimated as a limit:

$$A \approx \lim_{N \to \infty} \sum_{i=1}^{N} f\left(x_i^*\right) \Delta x \text{ on } x \in [a\ ,\ b]$$

where $\Delta x = \frac{b-a}{N}$ and $f\left(x_i^*\right)$ was the function value at left, right, or midpoints, x_i .

Now, we are going to be a bit more general in the sense that these rectangles no longer need to be equally spaced.
So think of the x-line like this:



Now, each Δx is different: $\Delta x_i = x_i - x_{i-1}$

Example: $\Delta x_1 = x_1 - x_0$

But our sum is still handled essentially the same way:

$$A \approx \sum_{i=1}^{N} f\left(x_i^*\right) \Delta x_i \text{ on } x \in [a\ ,\ b] = [x_0\ ,\ x_N] \text{ (no limit yet!)}$$

Where $\Delta x_i = x_i - x_{i-1}$ and $f\left(x_i^*\right)$ is the function evaluated at the left, right, or midpoint as necessary.

This particular version of calculating area is called the Riemann Sum.

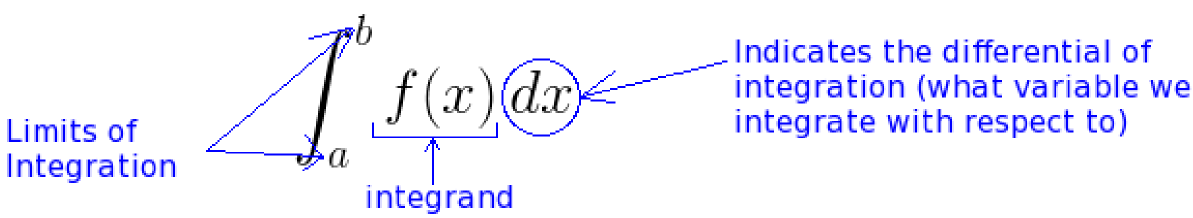
Definition of **<u>Definite Integral</u>** p. 263 Key Concept

If f is a function defined on $[a\ ,\ b]$, the definite integral of f from a to b is the number:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x \to 0}\left[\sum_{i=1}^{N} f\left(x_i^*\right)\Delta x_i\right]$$

provided that this limit exists. If it does exist, we say that f is <u>integrable</u> on $[a\ ,\ b]$.

Notation and Language:



Theorem: If f is continuous on $[a\ ,\ b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a\ ,\ b]$. That means $\int_a^b f(x)\,dx$ exists.

Example: Write $\lim_{N\to\infty}\sum_{i=1}^{N} x_i \sin\left(x_i\right)\Delta x_i$ on $[0\ ,\ \pi]$ as a definite integral. (DO NOT SOLVE)

$$\lim_{N\to\infty}\sum_{i=1}^{N} x_i \sin\left(x_i\right)\Delta x_i = \int_0^{\pi}\left(x\sin(x)\right)dx$$

Evaluating a Riemann Sum

Need to Know:

1. $$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$

For Example: $\sum_{k=1}^{5} k = 1+2+3+4+5 = 15$

Or: $\sum_{k=1}^{5} k = \frac{5(5+1)}{2} = \frac{5(6)}{2} = \frac{30}{2} = 15$

2. $\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$

For Example: $\sum_{k=1}^{5} k^2 = 1^2+2^2+3^2+4^2+5^2 = 1+4+9+16+25 = 55$

Or: $\sum_{k=1}^{5} k^2 = \frac{5(5+1)(2\cdot5+1)}{6} = \frac{5(6)(11)}{6} = 55$

3. $\sum_{k=1}^{N} k^3 = \left(\frac{N(N+1)}{2}\right)^2$

For Example: $\sum_{k=1}^{5} k^3 = 1^3+2^3+3^3+4^3+5^3 = 1+8+27+64+125 = 225$

Or: $\sum_{k=1}^{5} k^3 = \left(\frac{5(5+1)}{2}\right)^2 = \left(\frac{5(6)}{2}\right)^2 = 15^2 = 225$

4. $\sum_{k=1}^{N} c = Nc$

For Example: $\sum_{k=1}^{5} 3 = 3+3+3+3+3 = 5(3) = 15$

5. $\sum_{k=1}^{N} c\, a_i = c \sum_{k=1}^{N} a_i$

For Example: $\sum_{k=1}^{5} 3k = 3(1)+3(2)+3(3)+3(4)+3(5) = 3(1+2+3+4+5) = 45$

6. $\sum_{k=1}^{N} (a_i+b_i) = \sum_{k=1}^{N} a_i + \sum_{k=1}^{N} b_i$

7. $\sum_{k=1}^{N} (a_i-b_i) = \sum_{k=1}^{N} a_i - \sum_{k=1}^{N} b_i$

Example pg. 273 #20) Use $\int_a^b f(x)\,dx = \lim_{N\to 0}\left[\sum_{i=1}^{N} f\left(x_i^*\right)\Delta x_i\right]$ to evaluate $\int_1^4 \left(x^2+2x-5\right)dx$

Think of $f(x) = x^2+2x-5$ and $[a\ ,\ b] = [1\ ,\ 4]$

Now, find $\Delta x = \frac{b-a}{N} = \frac{4-1}{N} = \frac{3}{N}$ (because we do not know *N*)

We need to generalize x_i assuming an evenly space partition. So: $x_i = a+i\,\Delta x$

$$x_i = 1+i\left(\frac{3}{N}\right)$$

At this point, we just stuff everything we know into the form:

$$\int_a^b f(x)\,dx = \lim_{N\to 0}\left[\sum_{i=1}^{N} f\left(x_i^*\right)\Delta x_i\right]$$

So: $\int_1^4 \left(x^2+2x-5\right)dx = \lim_{N\to\infty}\sum_{i=1}^{N}\left(x_i^2+2x_i-5\right)\Delta x$

$$= \lim_{N\to\infty}\sum_{i=1}^{N}\left\{\left[\left(1+i\left(\frac{3}{N}\right)\right)^2+2\left(1+i\left(\frac{3}{N}\right)\right)-5\right]\left(\frac{3}{N}\right)\right\}$$

Expand $$= \lim_{N\to\infty}\frac{3}{N}\sum_{i=1}^{N}\left[1+\frac{6i}{N}+\frac{9i^2}{N^2}+2+\frac{6i}{N}-5\right]$$

Combine like terms $$= \lim_{N\to\infty}\frac{3}{N}\sum_{i=1}^{N}\left[\frac{9}{N^2}i^2+\frac{12}{N}i-2\right]$$

Distribute the sum $$= \lim_{N\to\infty}\frac{3}{N}\left[\frac{9}{N^2}\sum_{i=1}^{N}(i^2) + \frac{12}{N}\sum_{i=1}^{N}(i) - \sum_{i=1}^{N}(2)\right]$$

Use your rules $$= \lim_{N\to\infty}\frac{3}{N}\left[\frac{9}{N^2}\cdot\frac{N(N+1)(2N+1)}{6} + \frac{12}{N}\cdot\frac{N(N+1)}{2} - 2N\right]$$

Distribute $$= \lim_{N\to\infty}\left[\frac{27}{N^3}\cdot\frac{N(N+1)(2N+1)}{6} + \frac{36}{N^2}\cdot\frac{N(N+1)}{2} - \frac{6N}{N}\right]$$

Algebra $$= \lim_{N\to\infty}\left[\frac{9(N+1)(2N+1)}{2N^2} + \frac{18(N+1)}{N} - 6\right]$$

Reorganize $$= \lim_{N\to\infty}\left[\frac{9(2N^2+3N+1)}{2N^2} + \frac{18(N+1)}{N} - 6\right]$$

Simplify $= \lim_{N \to \infty} \left[9 + \frac{27}{2N} + \frac{9}{2N^2} + 18 + \frac{18}{N} - 6 \right]$

Evaluate Limit $= 9+0+0+18+0-6$

$= \mathbf{21}$

Theorem Midpoint Rule (pg. 268) A particular Riemann Sum

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{N} f\left(\overline{x_i}\right)\Delta x = \Delta x\left[f\left(\overline{x_1}\right) + f\left(\overline{x_2}\right) + f\left(\overline{x_3}\right) + \ldots + f\left(\overline{x_N}\right)\right]$$

where $\Delta x = \frac{b-a}{N}$ (a regularly spaced partition)

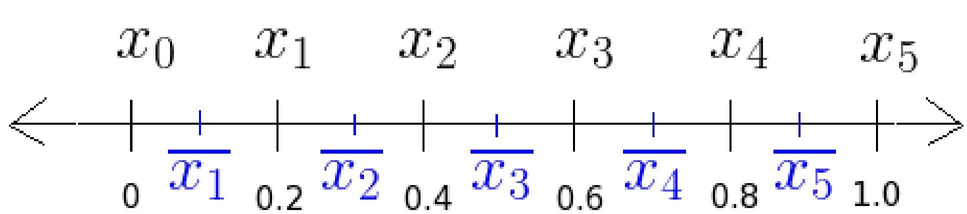
and $\overline{x_i} = \frac{1}{2}\left(x_{i-1} + x_i\right)$ which is the midpoint of $\left[x_{i-1}, x_i\right]$ any sub-interval

Example pg. 273 #13) Use Mid Point Rule to approximate $\int_0^1 \sin(x^2)\,dx$ with $N = 5$.

So, you need to calculate $\sum_{i=1}^{5} f\left(\overline{x_i}\right)\Delta x$

$$\Delta x = \frac{b-a}{N} = \frac{1-0}{N} = \frac{1-0}{5} = \frac{1}{5}$$

Now you need all of the midpoints.



$$\overline{x_1} = \frac{1}{2}(x_0+x_1) = \frac{1}{2}\left(0+\frac{1}{5}\right) = \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{10} = 0.1$$

$$\overline{x_2} = \frac{1}{2}(x_1+x_2) = \frac{1}{2}\left(\frac{1}{5}+\frac{2}{5}\right) = \frac{1}{2}\left(\frac{3}{5}\right) = \frac{3}{10} = 0.3$$

$$\overline{x_3} = \frac{1}{2}(x_2+x_3) = \frac{1}{2}\left(\frac{2}{5}+\frac{3}{5}\right) = \frac{1}{2}\left(\frac{5}{5}\right) = \frac{5}{10} = 0.5$$

$$\overline{x_4} = \frac{1}{2}(x_3+x_4) = \frac{1}{2}\left(\frac{3}{5}+\frac{4}{5}\right) = \frac{1}{2}\left(\frac{7}{5}\right) = \frac{7}{10} = 0.7$$

$$\overline{x_5} = \frac{1}{2}(x_4+x_5) = \frac{1}{2}\left(\frac{4}{5}+\frac{5}{5}\right) = \frac{1}{2}\left(\frac{9}{5}\right) = \frac{9}{10} = 0.9$$

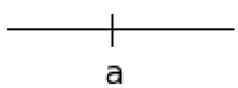
$$A \approx \sum_{i=1}^{5} f(\overline{x_i})\Delta x = \Delta x \sum_{i=1}^{5} f(\overline{x_i})$$

$$A \approx \frac{1}{5}\left(\sin(0.1^2)+\sin(0.3^2)+\sin(0.5^2)+\sin(0.7^2)+\sin(0.9^2)\right)$$

Then smash into calculator to get $A \approx 0.3084$

Now, just handling the integral notation without all the summing (short-cut methods)

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$



No area under a single point

2. If $a = b$, then $\Delta x = 0$, so $\int_a^b f(x)\,dx = 0$

3. $\int_a^b c\,dx = c(b-a)$ where c is a constant

4. $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$ constants move out

5. $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ splitting sums

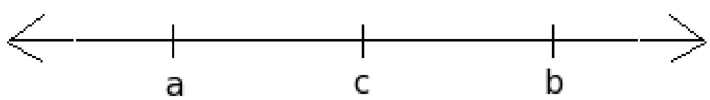
6. $\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$ and differences

Sometimes, even if we just know these simple properties we can quickly evaluate an integral, or at least break it into smaller chunks.

Example $\int_0^1 (5-4x^3)\,dx = \int_0^1 5\,dx - 4\int_0^1 x^3\,dx$

More Properties

7. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$ Intervals



$[a\,,\,c] + [c\,,\,b] = [a\,,\,b]$

8. If $f(x) \geq 0$ on $x \in [a\,,\,b]$ then $\int_a^b f(x)\,dx \geq 0$

   If you have a positive value function, you will have a positive valued area under the curve.

9. If $f(x) \geq g(x)$ on $x \in [a\,,\,b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

   If *f* is always bigger then *g*, then the area under *f* is always bigger than the area under *g*.

10. If $m \leq f(x) \leq M$ on $x \in [a\,,\,b]$ then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

    Same as $\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$

    (same as #9, it's just now you are bounding with constants)

Example pg. 274 #32) Evaluate the integral by interpreting it in terms of areas.

$$\int_{-2}^{2} \sqrt{4-x^2}\, dx$$

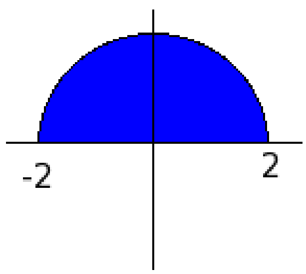
You want the area under the curve between -2 and +2

$$f(x) = \sqrt{4-x^2}$$
$$y = \sqrt{4-x^2}$$
$$y^2 = 4-x^2$$
$$x^2+y^2 = 4$$

So, we have circle center at (0, 0) with R = 2.

Recall: $\left(x-x_0\right)^2+\left(y-y_0\right)^2 = R^2$

Center: $\left(x_0 , y_0\right)$
radius: R



So, I need ½ the area of this circle

$$A_c = \pi r^2$$
$$\frac{1}{2} A_c = \frac{1}{2}\pi r^2$$
$$= \frac{1}{2}\pi (2)^2$$
$$= \frac{1}{2}\pi 4$$
$$= 2\pi$$

Notation Exercise:

Example If $\int_0^5 f(x)\, dx = 24$ and $\int_0^5 g(x)\, dx = 3$, find $\int_0^5 \left(2 f(x)-5 g(x)\right) dx$.

$$\int_0^5 \left(2 f(x)-5 g(x)\right) dx = 2\int_0^5 f(x)\, dx \; - \; 5\int_0^5 g(x)\, dx$$
$$= 2(24)-5(3)$$
$$= 48-15$$
$$= 33$$

Find: $\int_5^0 4 g(x)\, dx$

$$\int_5^0 4 g(x)\, dx = -4\int_0^5 g(x)\, dx = -4(3) = -12$$