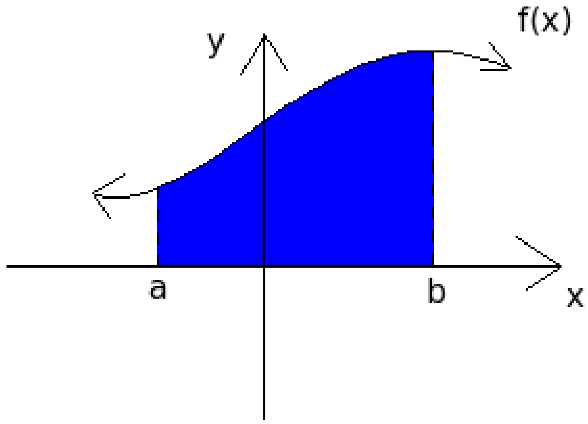
Section 5.1 Areas and Distances

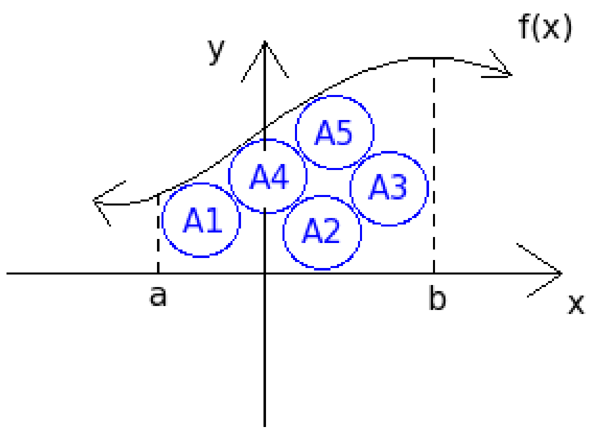
The Area Problem:



How might we evaluate the area (represented by the blue shaded region) between a curve and the x-axis between two points [*a* , *b*]?

We know how to find areas of circles, rectangles, trapezoids, triangles, etc. But what about irregularly shaped regions? What do we do then?

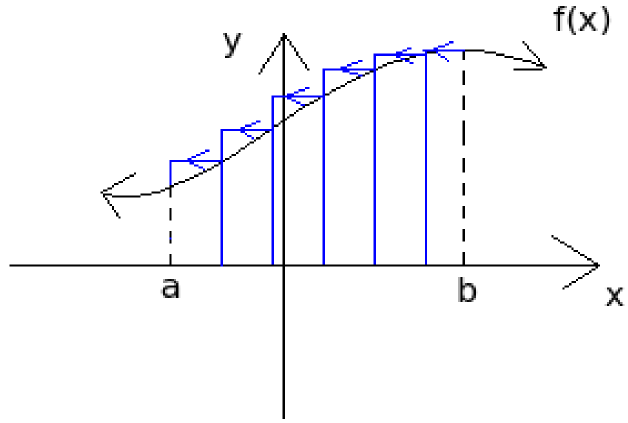
Idea: We can take some shape, circles, for example and keep filling the shaded area and adding up those known areas.



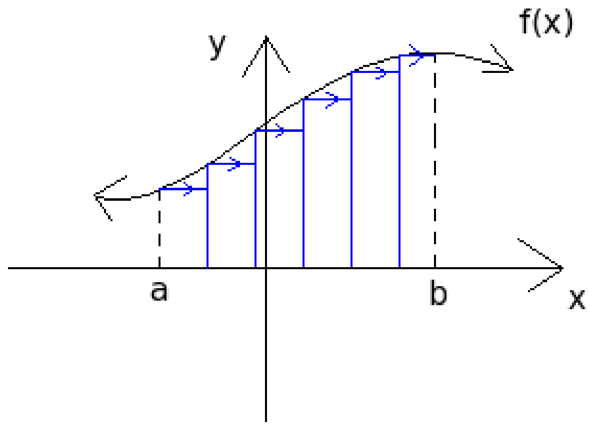
But that leaves lots of bits that I can't squeeze a circle into...doesn't it? That leads to <u>error</u>.

What might be a better shape to try?

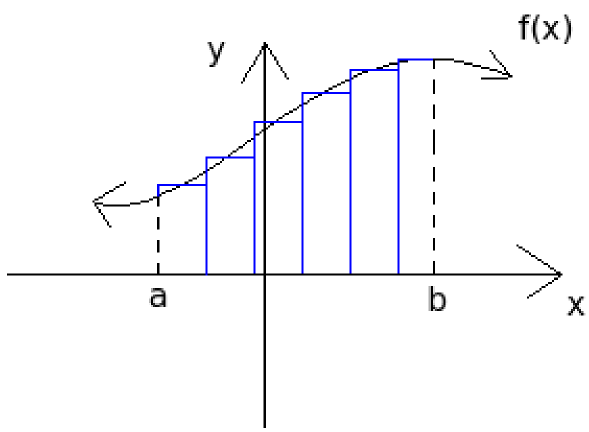
General? Which way might be best and why? (It depends on f(x)!)



<u>Option 1</u>
Line up right edge and draw rectangle to the left.



<u>Option 2</u>
Line up left edge and draw rectangle to the right.



<u>Option 3</u>
Line up on a midpoint and draw a rectangle outward in both directions.

In any of these options, I would add up the individual areas of the rectangles and could make a decent estimate of the area under $f(x)$.

How could I make a better estimate? For any of these cases, just use more rectangles. In fact, if I let $S_k =$ area of a single rectangle k, then:

Approximate Area Under a Curve $= \sum_{k=1}^{N} S_k = S_1 + S_2 + S_3 + S_4 + \ldots + S_n$

If I want an even better estimate, I could let $N \to \infty$

How would I write that?

Approximate Area Under a Curve $= \lim_{N \to \infty} \sum_{k=1}^{N} S_k = \lim_{N \to \infty} R_N$

The textbook lets $R_N = \sum_{k=1}^{N} S_k$ and calls R_N a Riemann Sum.

Some helpful simplifications not listed in your textbook:

1. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ the first *n* integers

   For Example: $\sum_{k=1}^{5} k = 1+2+3+4+5 = 15$

   Or: $\sum_{k=1}^{5} k = \frac{5(5+1)}{2} = \frac{5(6)}{2} = \frac{30}{2} = 15$

   You can see that as *n* gets bigger the formula is easier to manage than that long sum.

2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ the first *n* squares

   For Example: $\sum_{k=1}^{5} k^2 = 1^2+2^2+3^2+4^2+5^2 = 1+4+9+16+25 = 55$

   Or: $\sum_{k=1}^{5} k^2 = \frac{5(5+1)(2\cdot 5+1)}{6} = \frac{5(6)(11)}{6} = 55$

3. $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$ the first *n* cubes

   For Example: $\sum_{k=1}^{5} k^3 = 1^3+2^3+3^3+4^3+5^3 = 1+8+27+64+125 = 225$

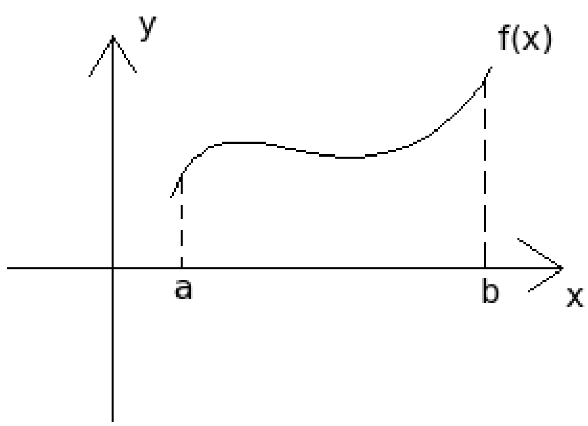
Or: $\sum_{k=1}^{5} k^3 = \left(\frac{5(5+1)}{2}\right)^2 = \left(\frac{5(6)}{2}\right)^2 = 15^2 = 225$

Example: Evaluate: $\sum_{k=1}^{4} (k^2 - 3k)$

$$\begin{aligned}\sum_{k=1}^{4} (k^2-3k) &= \sum_{k=1}^{4} k^2 - 3\sum_{k=1}^{4} k \\ &= \frac{4(4+1)(8+1)}{6} - 3\left(\frac{4(4+1)}{2}\right) \\ &= \frac{4(5)(9)}{6} - \frac{3(4)(5)}{2} \\ &= 30 - 30 \\ &= 0\end{aligned}$$

Returning to Riemann Sums

Let's think of a new function, *f(x)* and I want to approximate the area under the curve between $x = a$ and $x = b$.



Process:

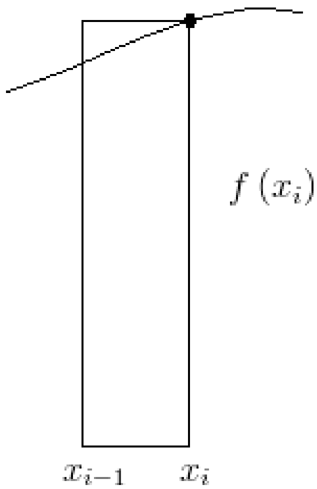
1. Divide the interval $[a\ ,\ b]$ into *n* equal chunks.

   How long is each chunk? $\frac{(b-a)}{n} = \frac{\text{total distance}}{\text{number of chunks}}$

2. Decide, based on Options 1, 2, and 3, for our rectangles, how we want to calculate area.

   For example, choose option 1. Right hand point on curve, drawing rectangles leftward.

   So, the area of an individual rectangle will be:

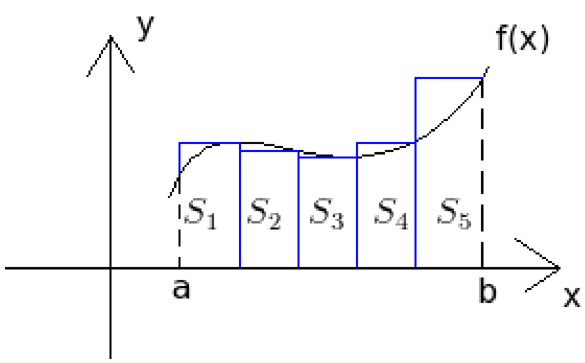


$$\frac{b-a}{n} = \Delta x$$

$$A = wl$$

$$A = \underbrace{\left(\frac{b-a}{n}\right)}_{\text{width } \Delta x} * \underbrace{f\left(x_i\right)}_{\text{function value at the right hand side}}$$

3. Calculate all the Areas that you need. Book Notation, find S_k

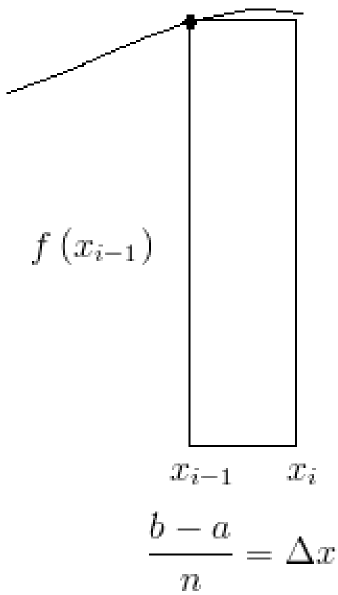


Find S_1 , S_2 , S_3 , S_4 , S_5

4. The, add them up. $R_n = S_1 + S_2 + S_3 + S_4 + S_5$.

<u>Alternatives</u>:

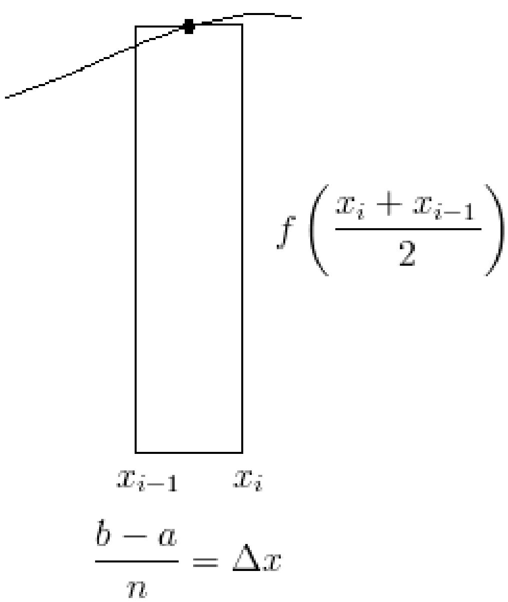
Change Step 2
If you wanted to use a different option:
Option 2: Left hand point on curve, drawing rectangles to the right.



$$A = wl$$

$$A = \underbrace{\left(\frac{b-a}{n}\right)}_{\text{width } \Delta x} * \underbrace{f\left(x_{i-1}\right)}_{\text{function value at the left hand side}}$$

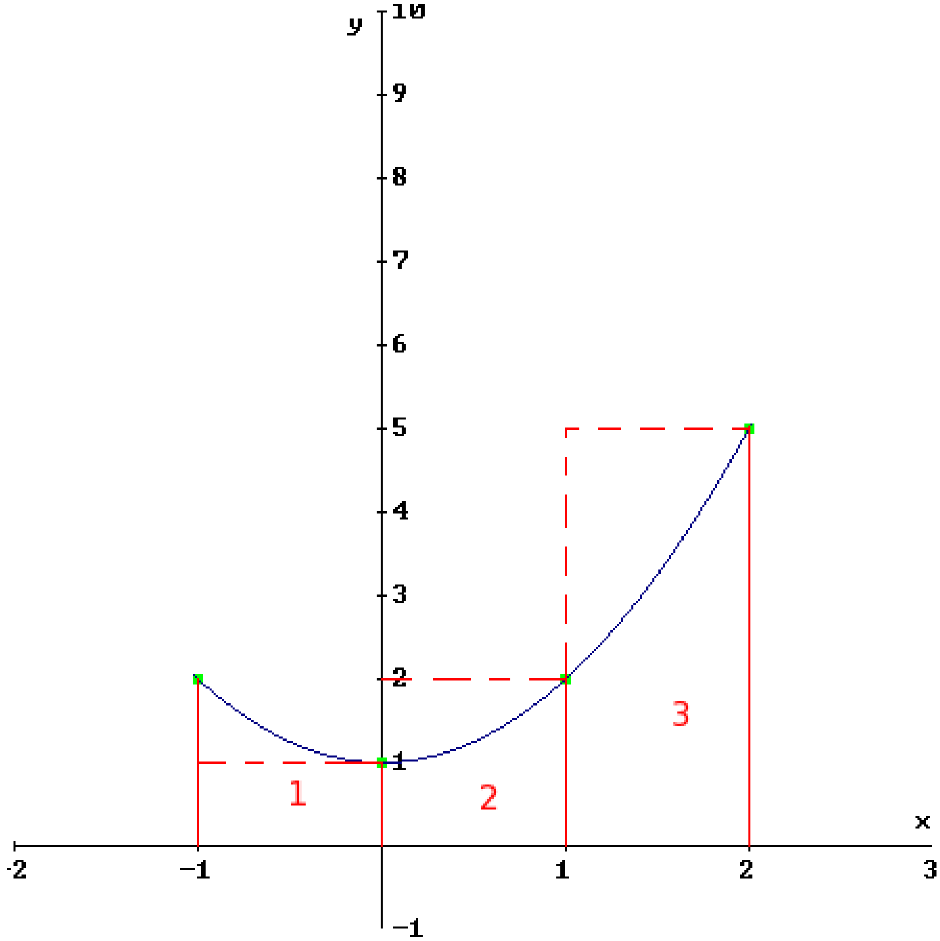
If you wanted to use Option 3. Midpoint then measured outward.



$$A = wl$$

$$A = \underbrace{\left(\frac{b-a}{n}\right)}_{\text{width } \Delta x} * \underbrace{f\left(\frac{x_i + x_{i-1}}{2}\right)}_{\text{function value at the midpoint}}$$

Example: Estimate the area under the graph of $f(x) = 1 + x^2$ from x = -1 to x = 2 using 3 rectangles and right endpoints.

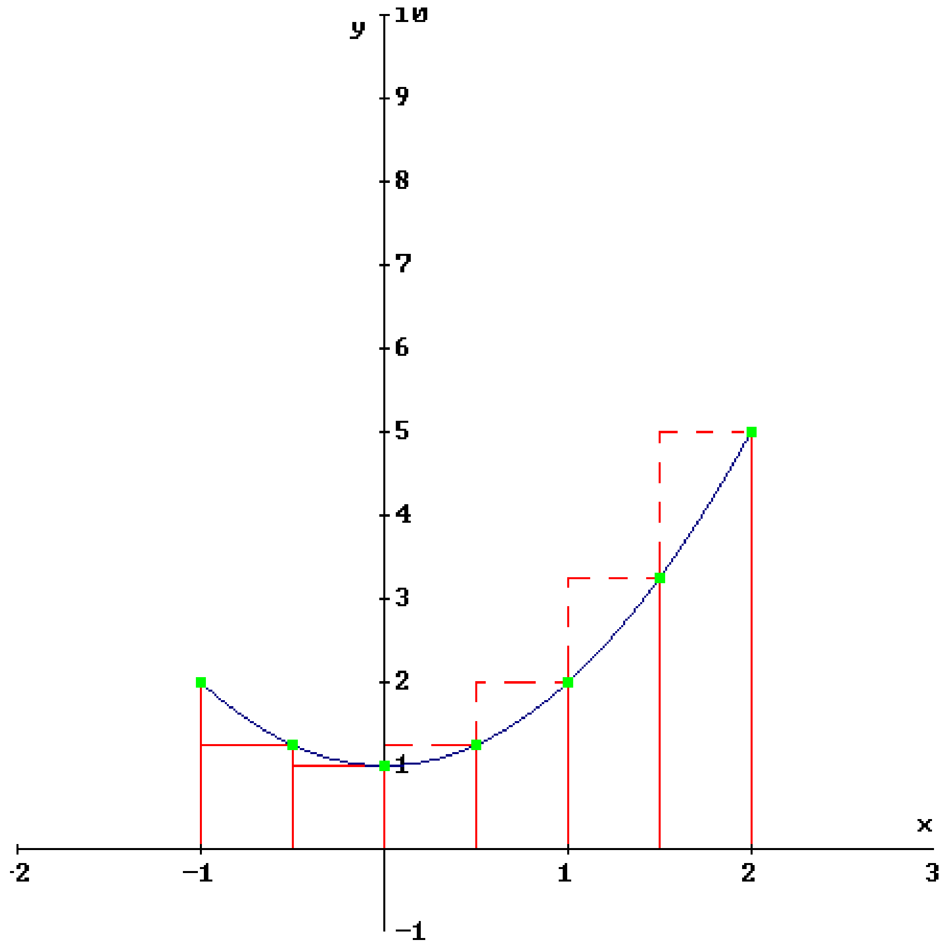


Note BIG over-estimate

$$\Delta x = \frac{2-(-1)}{3} = \frac{3}{3} = 1$$

$$S_1 = \Delta x f(0) = 1(1+0^2) = 1$$
$$S_2 = \Delta x f(1) = 1(1+1^2) = 2$$
$$S_3 = \Delta x f(2) = 1(1+2^2) = 5$$
$$R = S_1 + S_2 + S_3 = 1+2+5 = 8$$

Estimate with 6 rectangles (again an overestimate)



$$\Delta x = \frac{2-(-1)}{6} = \frac{3}{6} = \frac{1}{2}$$

$$S_1 = \Delta x f\left(\frac{-1}{2}\right) = \frac{1}{2}\left(1+\left(\frac{-1}{2}\right)^2\right) = \frac{1}{2}\left(\frac{5}{4}\right) = \frac{5}{8}$$

$$S_2 = \Delta x f(0) = \frac{1}{2}(1+0^2) = \frac{1}{2}$$

$$S_3 = \Delta x f\left(\frac{1}{2}\right) = \frac{1}{2}\left(1+\left(\frac{1}{2}\right)^2\right) = \frac{1}{2}\left(\frac{5}{4}\right) = \frac{5}{8}$$

$$S_4 = \Delta x f(1) = \frac{1}{2}(1+1^2) = 1$$

$$S_5 = \Delta x f\left(\frac{3}{2}\right) = \frac{1}{2}\left(1+\left(\frac{3}{2}\right)^2\right) = \frac{1}{2}\left(\frac{13}{4}\right) = \frac{13}{8}$$

$$S_6 = \Delta x f(2) = \frac{1}{2}(1+2^2) = \frac{5}{2}$$

$$R = S_1+S_2+S_3+S_4+S_5 = \frac{5}{8}+\frac{1}{2}+\frac{5}{8}+1+\frac{13}{8}+\frac{5}{2}$$

$$= \frac{5}{8}+\frac{4}{8}+\frac{5}{8}+\frac{8}{8}+\frac{13}{8}+\frac{20}{8} = \frac{55}{8} = 6\frac{7}{8} = 6.875$$

Side Note: Actual Area $= \int_{-1}^{2} \left(1+x^2\right) dx = x + \frac{x^3}{3}\Big|_{x=-1}^{x=2} = 2+\frac{8}{3}-\left(-1-\frac{1}{3}\right) = 6$

Example: Find an expression for the area under the graph *f* so that it is expressed as a limit. Do not evaluate.

$$f(x) = \frac{\ln x}{x} \quad \text{for } x \in [3\ ,\ 10]$$

$$A \approx \lim_{N\to\infty} \sum_{k=1}^{N} S_k$$ Remember $S_k =$ area of each rectangle.

Choose S_k to be represented by right hand end point rectangles, then:

$$S_k = \Delta x f\left(x_k\right) \quad k = 1,\ 2,\ 3,\ \ldots,\ N$$

$$\Delta x = \frac{b-a}{N} = \frac{10-3}{N} = \frac{7}{N}$$

$$f\left(x_k\right) = \frac{\ln x_k}{x_k}$$

So: $A \approx \lim_{N\to\infty} \sum_{k=1}^{N} \frac{7}{N} \cdot \frac{\ln x_k}{x_k}$

Note: $\begin{aligned} x_k &= 3+k\,\Delta x \\ &= 3+\frac{7k}{N} \end{aligned}$ you should define this to be clear.