Section 4.7 Anti-Derivatives

Essentially, given a derivative, we want to find the function that produced it (you might want to think of this as doing derivatives backwards).

Definition A function F is called an anti-derivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Are anti-derivatives unique? NO

Think of $f(x) = 2\text{x}$

What function, *F(x)*, did this come from?

$$F'(x) = 2\text{x} \Rightarrow F(x) = x^2$$

i.e. OR $F(x) = x^2 + 4$

OR $F(x) = x^2 - 7$

There are of course countless other *F(x)*. Notice, however, that these functions F are all similar —their differences only being a constant.

Theorem If F is an anti-derivative of f on an interval I, then the most general anti-derivative of f on I is

$$F(x) + c$$

where c is an arbitrary constant.

In general, then, how do we puzzle out anti-derivatives? The come directly from the derivative rules that <u>we</u> <u>already</u> <u>know</u>!

Example Derivative Rule

function	derivative
x^n	$n\,x^{n-1}$

Now we go backwards...

Anti-derivative Rules

anti-derivative		derivative
$\frac{1}{n+1} x^{n+1}$	$\leftarrow$	$x^n \quad (n \neq 1)$
$\ln\lvert x\rvert$	$\leftarrow$	$\frac{1}{x}$
e^x	$\leftarrow$	e^x
$-\cos x$	$\leftarrow$	$\sin x$
$\sin x$	$\leftarrow$	$\cos x$
$\sec x$	$\leftarrow$	$\sec x \tan x$
$\tan x$	$\leftarrow$	$\sec^2 x$

Don't Forget “+c”!

Likewise, all of the general notions of sums and differences are maintained.

anti-derivative		derivative
$c\,f(x)$	$\leftarrow$	$c\,F(x)$
$f(x)+g(x)$	$\leftarrow$	$F(x)+G(x)$

Example Find a general anti-derivative for:

pg. 246 #2) $f(x) = 1-x^3+12x^5$

$$F(x) = x-\frac{1}{4}x^4+12\left(\frac{1}{6}\right)x^6+c$$

$$F(x) = x-\frac{1}{4}x^4+2x^6+c$$

Check:
$$F'(x) = 1-\frac{1}{4}\left(4x^3\right)+2\left(6x^5\right)+0$$
$$F'(x) = 1-x^3+12x^5$$

Example Find a general anti-derivative for:

pg. 246 #6)
$$f(x) = \sqrt[4]{x^3}+\sqrt[3]{x^4}$$
$$f(x) = x^{\frac{3}{4}}+x^{\frac{4}{3}}$$ simplify notation

$$F(x) = \frac{1}{\frac{7}{4}}x^{\frac{7}{4}}+\frac{1}{\frac{7}{3}}x^{\frac{7}{3}}+c$$

$$F(x) = \frac{4}{7}\sqrt[4]{x^7}+\frac{3}{7}\sqrt[3]{x^7}+c$$

$$F(x) = \frac{4}{7}x\sqrt[4]{x^3}+\frac{3}{7}x^2\sqrt[3]{x}+c$$

Check:
$$F'(x) = \frac{4}{7}\left(\frac{7}{4}\right)x^{\frac{3}{4}} + \frac{3}{7}\left(\frac{7}{3}\right)x^{\frac{4}{3}}+0$$
$$F'(x) = x^{\frac{3}{4}}+x^{\frac{4}{3}} = f(x)$$

Example Find the anti-derivative *F* of *f* that satisfies the given condition.

pg. 246 # 14)
$$f(x) = 4-3(1+x^2)^{-1} \qquad F(1) = 0$$
$$f(x) = 4-\frac{3}{1+x^2}$$

$F(x) = 4x-3\tan^{-1}(x)+c$ note: $\tan^{-1}(x) = \text{atan}(x) = \text{arctan}(x);\ \tan^{-1}(x) \neq \frac{1}{\tan(x)}$

So, we have the anti-derivative, but we now must satisfy the condition $F(1)=0$

$$F(1) = 4(1)-3\tan^{-1}(1)+c = 0$$
$$4-3\left(\frac{\pi}{4}\right)+c = 0$$
$$\frac{16}{4}-\frac{3\pi}{4} = -c$$
$$\frac{16-3\pi}{4} = -c$$
$$\Rightarrow \ c = \frac{3\pi-16}{4}$$

So: $F(x) = 4x-3\tan^{-1}(x)+\frac{3\pi-16}{4}$

** In this case, then, our constant *c* is very specific to the problem.

Check:
$$F'(x) = 4-3\left(\frac{1}{1+x^2}\right)+0$$
$$F'(x) = 4-\frac{3}{1+x^2} = f(x)$$

Example Find f.

pg. 246 #23) $f''(x) = 24x^2+2x+10 \qquad f(1) = 5 \quad f'(1) = -3$

$$f'(x) = 24\left(\frac{1}{3}\right)x^3+2\left(\frac{1}{2}\right)x^2+10x+c$$
$$f'(x) = 8x^3+x^2+10x+c$$
apply condition $f'(1) = 8(1)^3+(1)^2+10(1)+c = -3$
$$8+1+10+3 = -c$$
$$-22 = c$$

$$f'(x) = 8x^3 + x^2 + 10x - 22$$

$$f(x) = 8\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)x^3 + 10\left(\frac{1}{2}\right)x^2 - 22x + d$$

$$f(x) = 2x^4 + \frac{1}{3}x^3 + 5x^2 - 22x + d$$

apply condition: $f(1) = 2(1)^4 + \frac{1}{3}(1)^3 + 5(1)^2 - 22(1) + d = 5$

$$2 + \frac{1}{3} + 5 - 22 - 5 = -d$$

$$\frac{1}{3} - 20 = -d$$

$$\frac{-59}{3} = -d$$

$$d = \frac{59}{3}$$

$$f(x) = 2x^4 + \frac{1}{3}x^3 + 5x^2 - 22x + \frac{59}{3}$$

$$f(1) = 2(1)^4 + \frac{1}{3}(1)^3 + 5(1)^2 - 22(1) + \frac{59}{3} = 5$$

Check: $f'(x) = 8x^3 + x^2 + 10x - 22$

$$f'(1) = 8(1)^3 + (1)^2 + 10(1) - 22 = -3$$

$$f''(x) = 24x^2 + 2x + 10$$

Example: A particle is moving with the given data. Find the position of the particle.

$v(t) = 1.5\sqrt{t}$ $\quad$ $s(4) = 10$

$$v(t) = \frac{3}{2}t^{\frac{1}{2}}$$

$$s(t) = \frac{3}{2}\left(\frac{1}{\frac{3}{2}}\right)t^{\frac{3}{2}} + c$$

$$s(t) = \frac{3}{2}\left(\frac{2}{3}\right)t^{\frac{3}{2}} + c$$

$$s(t) = t^{\frac{3}{2}} + c$$

apply condition: $s(4) = 4^{\frac{3}{2}} + c = 0$

$s(4) = \sqrt{64} + c = 10$

$8 + c = 10 \quad \Rightarrow \quad c = 2$

So: $s(t) = t^{\frac{3}{2}} + 2$