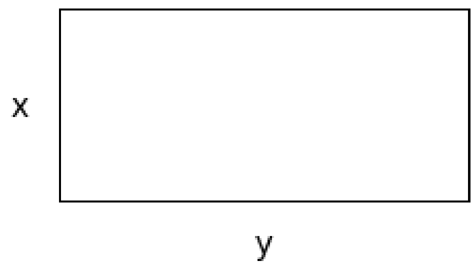
Section 4.5 Optimization Problems

This section is dedicated to word problems where we need to find maximum or minimum.

Example pg. 232 #6) Find the dimensions of a rectangle with an area of 1000 m^2 whose perimeter is as small as possible.

1. Draw a picture and label as necessary



A = 1000 m^2
P = minimize

2. Develop equations

$$A = xy = 1000$$

$$P = 2x + 2y$$

3. Write the equation you want to minimize/maximize in terms of one variable

Use area $x = \frac{1000}{y}$ OR $y = \frac{1000}{x}$

Stick in perimeter
$$P = 2x + 2\left(\frac{1000}{x}\right)$$
$$P(x) = 2x + \frac{2000}{x}$$

4. Find 1st derivative and critical points

$$P'(x) = 2 + \frac{2000(-1)}{x^2}$$
$$= \frac{2x^2 - 2000}{x^2}$$
$$= \frac{2(x^2 - 1000)}{x^2}$$

Critical Points:

$$P'(x) = 0 \qquad\qquad P'(x) = \text{DNE}$$
$$x^2 - 1000 = 0 \qquad\qquad x^2 = 0$$
$$x = \pm\sqrt{1000} \qquad\qquad x = 0$$

Throw out negative lengths   Not Physical - You'd have a line

So, $x = \sqrt{1000}$

5. Test critical points and end points (if any) for maximum and minimum

$$P''(x) = \frac{x^2(4x) - (2x^2 - 2000)(2x)}{\left(x^2\right)^2}$$
$$= \frac{\cancel{4x^3} - \cancel{4x^3} + 4000x}{x^4}$$
$$P''(x) = \frac{4000}{x^3}$$

$P''\left(\sqrt{1000}\right) = \frac{+}{+} = + \quad \Rightarrow \quad x = +\sqrt{1000}$ is a critical point

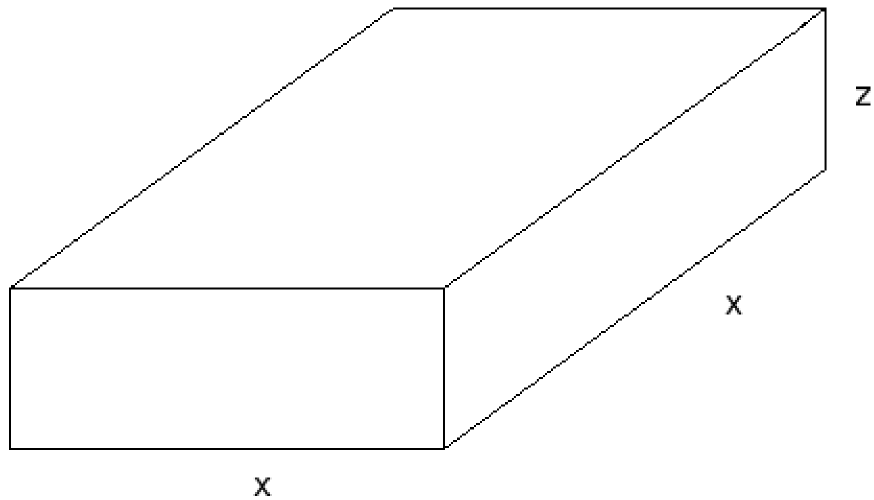
6. Find other variable

$$y = \frac{1000}{x} = \frac{1000}{\sqrt{1000}} = \sqrt{1000}$$

Our intuition is correct that the rectangle is in fact, a square with dimensions $\sqrt{1000}$ x $\sqrt{1000}$ m

Example pg. 232 #10) A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

1. Draw a picture and label as necessary



V = 32,000 cm^3 , P = minimize

2. Develop equations

Volume $V = x \cdot x \cdot z = x^2 z = 32{,}000$
Surface Area $SA = 2x \cdot z + 2x \cdot z + x \cdot x \leftarrow$ Open Top, only one
$SA = 4xz + x^2$

3. Write the equation you want to minimize/maximize in terms of one variable

$$x^2 z = 32{,}000$$

$$z = \frac{32{,}000}{x^2} \quad \text{OR} \quad x = +\sqrt{\frac{32{,}000}{z}}$$

Stick in Surface Area

$$SA(x) = 4x\left(\frac{32{,}000}{x^2}\right) + x^2$$

$$= \frac{128{,}000}{x} + x^2$$

4. Find 1st derivative and critical points

$$SA'(x) = \frac{128{,}000(-1)}{x^2} + 2x$$
$$= \frac{-128{,}000 + 2x^3}{x^2}$$

Critical Points:

$$SA'(x) = 0 \qquad \text{OR} \qquad SA'(x) = \text{DNE}$$
$$2x^3 - 128{,}000 = 0 \qquad\qquad x^2 = 0$$
$$2(x^3 - 64{,}000) = 0 \qquad\qquad x = 0$$
$$\text{Recall } (x^3 - a^3) = (x^2 + ax + a^3) \qquad\qquad \text{Not Physical}$$
$$2(x-40)(x^2 - 40x + 1600) = 0$$
$$x - 40 = 0 \qquad x^2 + 40x + 1600 = 0$$
$$x = 40 \qquad \text{Imaginary Roots}$$

5. Test critical points and end points (if any) for maximum and minimum

$$SA''(x) = \frac{x^2(6x^2) - (2x^3 - 128{,}000)(2x)}{\left(x^2\right)^2}$$
$$= \frac{6x^4 - 4x^4 + 256{,}000x}{x^4}$$
$$= \frac{2x^4 + 256{,}000x}{x^4}$$
$$SA''(x) = \frac{x^3 + 128{,}000}{x^3}$$

Test: $SA''(40) = \frac{+}{+} = + \quad \Rightarrow \quad x = 40$ is a min. for SA

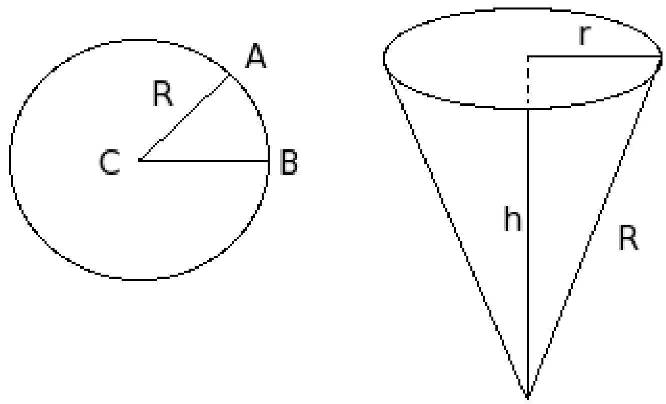
6. Find other variable

$$z = \frac{32{,}000}{x^2} = \frac{32{,}000}{40^2} = \frac{32{,}000}{1{,}600} = 20$$

So, our box has a 40 x 40 cm^2 base and a height of 20 cm.

Example pg. 233 #23) A cone shaped drinking cup is made from a circular piece of paper of radius *R* by cutting out a sector and joining the edges *CA* and *CB*. Find the maximum capacity of such a cup.

1. Draw a picture and label as necessary



Maximize the volume of the cup!

2. Develop equations

Volume of a right-circular cone: $V = \frac{\pi}{3} r^2 h$

Pythagorean Theorem: $r^2 + h^2 = R^2$

3. Write the equation you want to minimize/maximize in terms of one variable (note: *R* is known and is not a variable)

$h^2 = R^2 - r^2$ OR $r^2 = R^2 - h^2$

$h = \sqrt{R^2 - r^2}$

Stick in volume:
$$V = \frac{pi}{3}\left(R^2 - h^2\right) h$$
$$V(h) = \frac{\pi}{3} R^2 h - \frac{\pi}{3} h^3$$

4. Find 1st derivative and critical points

$$V'(h) = \frac{\pi}{3} R^2 - \frac{\pi}{3} \cdot 3h^2$$
$$V'(h) = \frac{\pi}{3} R^2 - \pi h^2$$

$$V'(h) = 0$$
$$\frac{\pi}{3}R^2 - \pi h^2 = 0$$
$$\frac{\pi}{3}R^2 = \pi h^2$$
Critical Points: $$\frac{1}{3}R^2 = h^2$$
$$\sqrt{\frac{1}{3}R^2} = h$$
$$h = \frac{1}{\sqrt{3}}R$$

5. Test critical points and end points (if any) for maximum and minimum

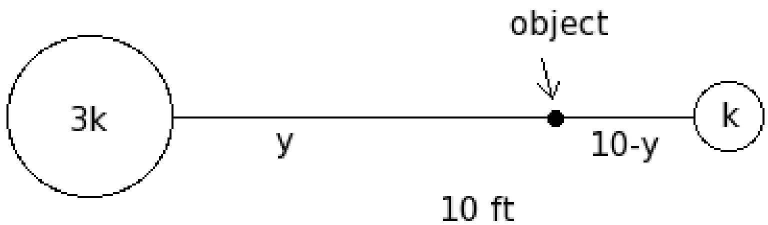
$$V''(h) = 0 - 2\pi h$$
$$V''(h) = -2\pi h$$
$$V''\left(\frac{1}{\sqrt{3}}R\right) = -2\pi\left(\frac{1}{\sqrt{3}}R\right) = (\text{-}) \quad \Rightarrow \quad \frac{1}{\sqrt{3}}R \text{ is a max for V}$$

6. Find other variable

$$V\left(\frac{1}{\sqrt{3}}R\right) = \frac{\pi}{3}R^2\left(\frac{1}{\sqrt{3}}R\right) - \frac{\pi}{3}\left(\frac{1}{\sqrt{3}}R\right)^3$$
$$= \frac{\pi}{3\sqrt{3}}R^3 - \frac{\pi}{3}\left(\frac{1}{3\sqrt{3}}\right)R^3$$
$$= \frac{\pi R^3}{3\sqrt{3}} - \frac{\pi R^3}{9\sqrt{3}} = \frac{3\pi R^3 - \pi R^3}{9\sqrt{3}}$$
$$V = \frac{2\pi R^3}{9\sqrt{3}} \quad \text{units}^3$$

Example pg. 234 #31) The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 ft apart, where should an object be placed on the line between the sources so as to receive the least illumination?

1. Draw a picture and label as necessary



$I \propto \frac{k}{y^2}$ I is illumination, k is the strength of the source, y is the distance from source

2. Develop equations

At my object:

$$I = \frac{3k}{y^2} + \frac{k}{(10-y)^2} \qquad y \in (0\ ,\ 10)$$

3. Write the equation you want to minimize/maximize in terms of one variable

I want to minimize I and it's already of one variable.

4. Find 1st derivative and critical points

$$I'(y) = \frac{3k(-2)}{y^3} + \frac{k(-2)}{(10-y)^3}(-1)$$

$$= \frac{-6k}{y^3} + \frac{2k}{(10-y)^3}$$

$$I'(y) = \frac{-6k(10-y)^3 + 2k(y^3)}{y^3(10-y)^3}$$

Find critical points:

$$-6k(10-y)^3 + 2k(y^3) = 0$$
$$2y^3 = 6(10-y)^3$$
$$y^3 = 3(10-y)^3$$
$$y = \sqrt[3]{3}(10-y)$$
$$y = 10\sqrt[3]{3} - \sqrt[3]{3}\,y$$
$$\left(1+\sqrt[3]{3}\right)y = 10\sqrt[3]{3}$$
$$y = \frac{10\sqrt[3]{3}}{1+\sqrt[3]{3}} \approx 5.905 \text{ ft} \leftarrow \text{unit from problem}$$

5. Test critical points and end points (if any) for maximum and minimum

$$I''(y) = \frac{y^3(10-y)^3\frac{d}{dy}\left(-6k(10-y)^3+2ky^3\right) - \left(-6k(10-y)^3+2ky^3\right)\frac{d}{dy}\left(y^3(10-y)^3\right)}{\left(y^3(10-y)^3\right)^2}$$

$$= \frac{y^3(10-y)3\left[-6k(3)(10-y)^2(-1)+6ky^2\right]}{y^6(10-y)^6}$$
$$- \frac{\left[-6k(10-y)^3+2ky^3\right]\left[y^3(3)(10-y)^2(-1)+(10-y)^3 3y^2\right]}{y^6(10-y)^6}$$

$$= \frac{y^3(10-y)^3\left[18k(10-y)^2+6ky^2\right] + \left[6k(10-y)^3-2ky^3\right]\left[-3y^3(10-y)^2+3y^2(10-y)^3\right]}{y^6(10-y)^6}$$

$$= \frac{y^3(10-y)^3\left[18k(100-20y+y^2)+6ky^2\right] + 3y^2(10-y)^2\left[6k(10-y)^3-2ky^3\right]\left[y+10-y\right]}{y^6(10-y)^6}$$

$$= \frac{y^3(10-y)^3\left[24ky^2-360ky+1800k\right] + 30y^2(10-y)^2\left[6k(10-y)^3-2ky^3\right]}{y^6(10-y)^6}$$

This is getting messy quick. Moving to calculator.

Check $I''(5.905)$

$$\approx \frac{(205)(69)(511)k+(30)(35)(17)(3)k}{(42{,}180)(4750)} = (+) \text{ provided } k > 0$$

$\Rightarrow y \approx 5.905$ is a minimum for I

So, the object should be placed about 5.905 ft from the larger light source.