Section 4.4 Curve Sketching Continued

One can be even more precise with curve sketching techniques by adding a few test steps. Specifically:

1. Check domain (which we already do)
2. Check intercepts *f(0)* and *f(x) = 0*
3. Check symmetry $\left(f(-x) = -f(x) \quad \text{odd}\,; \quad f(-x) = f(x) \quad \text{even}\right)$
4. Check asymptotes – vertical & horizontal
5. Then do I/D and *f ' (x)*
6. Local max & min
7. CU / CD on *f '' (x)*
8. Inflection points

Then...sketch

Example Sketch $y = \dfrac{x}{x^2-9} \quad \Rightarrow \quad f(x) = \dfrac{x}{x^2-9}$

1. Domain:

$$x^2-9 \neq 0$$
$$x \neq -3\ ,\ +3$$
$$x \in (-\infty\ ,\ -3) \cup (-3\ ,\ 3) \cup (3\ ,\ \infty)$$

2. Intercepts:

$$f(0) = \frac{0}{0-9} = 0 \quad \Rightarrow \quad (0\ ,\ 0)$$

3. Symmetry:

$$f(-x) = -\frac{x}{(-x)^2-9} = -\frac{x}{x^2-9} = -f(x)$$

So, our function is ODD. (symmetry about the origin)

4. Asymptotes:

Vertical Asymptote (denominator = 0):

$$x^2 - 9 = 0$$
$$(x+3)(x-3) = 0$$
$$x = -3 \quad x = +3$$

Horizontal Asymptote (rules)

$$y = 0$$

5. Now what we did last in section: I/D

$$f'(x) = \frac{(x^2-9)(1) - x(2x)}{(x^2-9)^2}$$

$$= \frac{x^2-9-2x^2}{(x^2-9)^2}$$

$$= \frac{-x^2-9}{(x^2-9)^2}$$

$$= \frac{-(x^2+9)}{(x^2-9)^2} = -\frac{(x^2+9)}{(x-3)^2(x+3)^2}$$

CP: $-(x^2+9) = 0$ imaginary $\qquad (x^2-9)^2 = 0$, $x = \pm 3$

6. CU / CD:

$$f''(x) = \frac{(x^2-9)^2(-2x) - \left(-(x^2+9)\right)(2)(x^2-9)(2x)}{\left((x^2-9)^2\right)^2}$$

$$= \frac{-2x(x^2-9)^2 + 4x(x^2-9)(x^2+9)}{(x^2-9)^4}$$

$$= \frac{2x(x^2-9)\left[-(x^2-9) + 2(x^2+9)\right]}{(x^2-9)^4}$$

Algebra!

$$= \frac{2x\left[-x^2+9+2x^2+18\right]}{(x^2-9)^3}$$

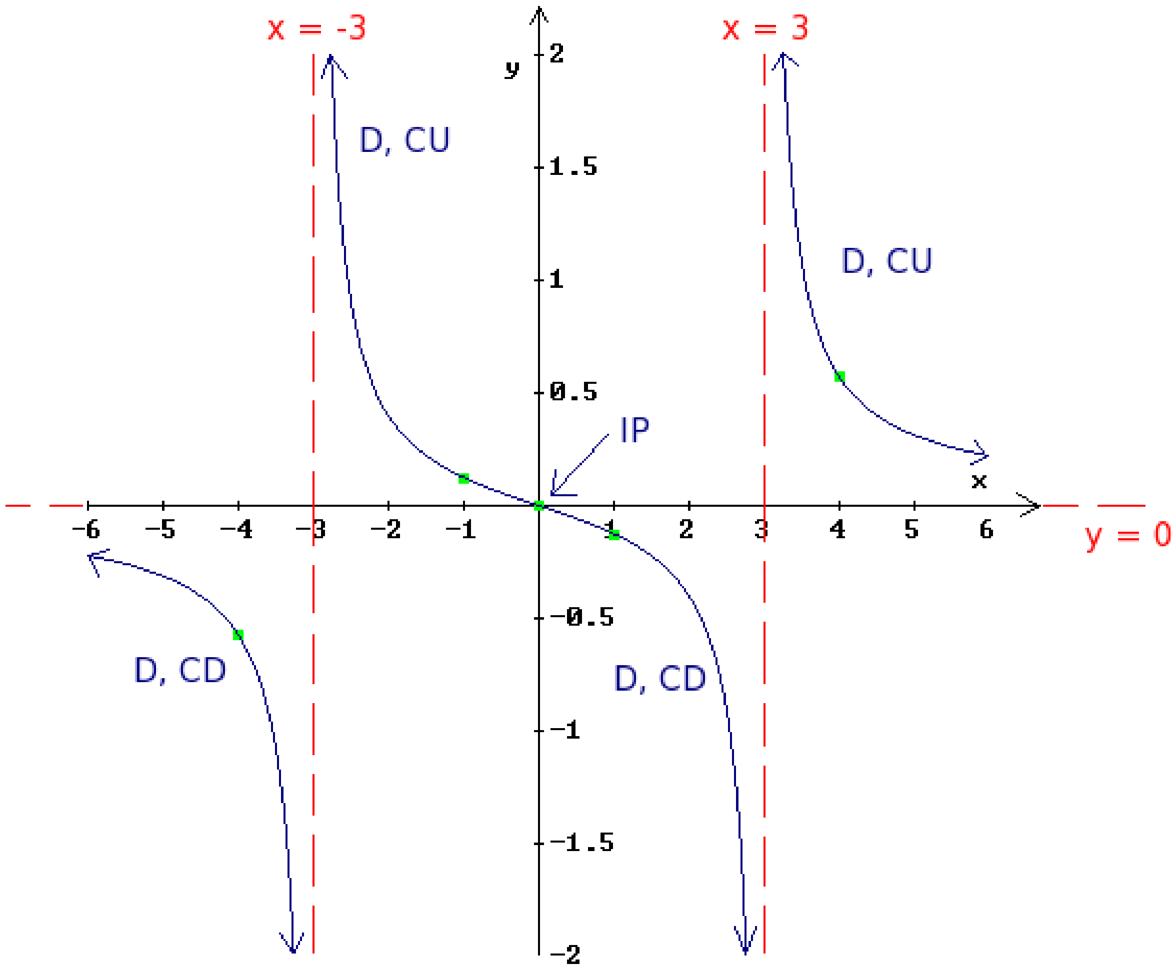
$$= \frac{2x\left(x^2+27\right)}{(x^2-9)^3}$$

Possible IP: $2x = 0$, $x = 0$ $\qquad x^2+27 = 0$ imaginary

7. Make a table:

Interval CP / IP	Test Point	$f'(x) = \frac{-(x^2+9)}{(x^2-9)^2}$	Increasing Decreasing	$f''(x) = \frac{2x(x^2+27)}{(x^2-9)^3}$	CU CD	$f(x) = \frac{x}{x^2-9}$
$(-\infty\ ,\ -3)$	-4	$\frac{(-)(+)}{(+)} = (-)$	Decreasing	$\frac{(-)(+)}{(+)} = (-)$	CD	$-\frac{4}{7}$
-3	-3	$\frac{(-)(+)}{(0)} = \text{DNE}$		$\frac{(-)(+)}{(0)} = \text{DNE}$		$-\frac{3}{0} = \text{DNE}$
$(-3\ ,\ 0)$	-1	$\frac{(-)(+)}{(+)} = (-)$	Decreasing	$\frac{(-)(+)}{(-)} = (+)$	CU	$\frac{-1}{-8} = \frac{1}{8}$
0	0	$\frac{(-)(+)}{(+)} = (-)$	Decreasing	$\frac{(0)(+)}{(-)} = (0)$	IP	$\frac{1}{-9} = 0$
$(0\ ,\ 3)$	1	$\frac{(-)(+)}{(+)} = (-)$	Decreasing	$\frac{(+)(+)}{(-)} = (-)$	CD	$\frac{1}{-8} = -\frac{1}{8}$
3	3	$\frac{(-)(+)}{(0)} = \text{DNE}$		$\frac{(+)(+)}{(0)} = \text{DNE}$		$\frac{3}{0} = \text{DNE}$
$(3\ ,\ \infty)$	4	$\frac{(-)(+)}{(+)} = (-)$	Decreasing	$\frac{(+)(+)}{(+)} = (+)$	CU	$\frac{4}{7}$

8. Sketch:



Example pg. 225, #7) Sketch $y = 2x^5 - 5x^2 + 1$

1. Domain

   Polynomial: $(-\infty\ ,\ \infty)$

2. Intercepts

   $f(0) = 2(0)^5 - 5(0)^2 + 1 = \quad \Rightarrow \quad (0\ ,\ 1)$

3. Symmetry

   $$f(-x) = 2(-x)^5 - 5(-x)^2 + 1$$
   $$= -2x^5 - 5x^2 + 1$$
   $$= -(2x^5 + 5x^2 - 1)$$

   Not even or odd

4. Asymptotes

   None

5. $f'(x)$

   $$f'(x) = 10x^4 - 10x$$
   $$= 10x(4x^3 - 1)$$
   $$= 10x(x-1)(x^2 + x + 1)$$

   CP: $\{0\ ,\ 1\}$

6. $f''(x)$

   $$f''(x) = 40x^3 - 10$$
   $$= 10(4x^3 - 1)$$
   $$= 10\left(\sqrt[3]{4}\,x - 1\right)\left(\left(\sqrt[3]{4}\,x\right)^2 + \left(\sqrt[3]{4}\,x\right) + 1\right)$$

   Possible IP:
   $$\sqrt[3]{4}\,x - 1 = 0$$
   $$\sqrt[3]{4}\,x = 1$$
   $$x = \frac{1}{\sqrt[3]{4}} \approx 0.62996$$

7. Make a Table

Interval CP / IP	Test Point	$f'(x) = 10x(x-1)(x^2+x+1)$	Increasing / Decreasing	$f''(x) = 10\left(\sqrt[3]{4}x-1\right)\left(\left(\sqrt[3]{4}x\right)^2+\left(\sqrt[3]{4}x\right)+1\right)$	CU / CD	$f(x) = 2x^5-5x^2+1$
$(-\infty\ ,\ 0)$	-1	(-)(-)(+) = (+)	Increasing	(+)(-)(+) = (-)	CD	$-2-5+1=-6$
0	0	(0)(-)(+) = 0	Maximum	(+)(-)(+) = (-)	CD	1
$\left(0\ ,\ \frac{1}{\sqrt[3]{4}}\right)$	$\frac{1}{2}$	(+)(-)(+) = (-)	Decreasing	(+)(-)(+) = (-)	CD	$\frac{2}{32}-\frac{5}{4}+1=-\frac{6}{32}$
$\frac{1}{\sqrt[3]{4}}$	$\frac{1}{\sqrt[3]{4}}$	(+)(-)(+) = (-)	Decreasing	(+)(0)(+) = 0	IP	-0.78583
$\left(\frac{1}{\sqrt[3]{4}}\ ,\ 1\right)$	$\frac{3}{4}$	(+)(-)(+) = (-)	Decreasing	(+)(+)(+) = (+)	CU	$\frac{-1370}{1024}$
1	1	(+)(0)(+) = 0	Minimum	(+)(+)(+) = (+)	CU	$2-5+1=-2$
$(1\ ,\ \infty)$	2	(+)(+)(+) = (+)	Increasing	(+)(+)(+) = (+)	CU	$64-20+1=45$

Aside:

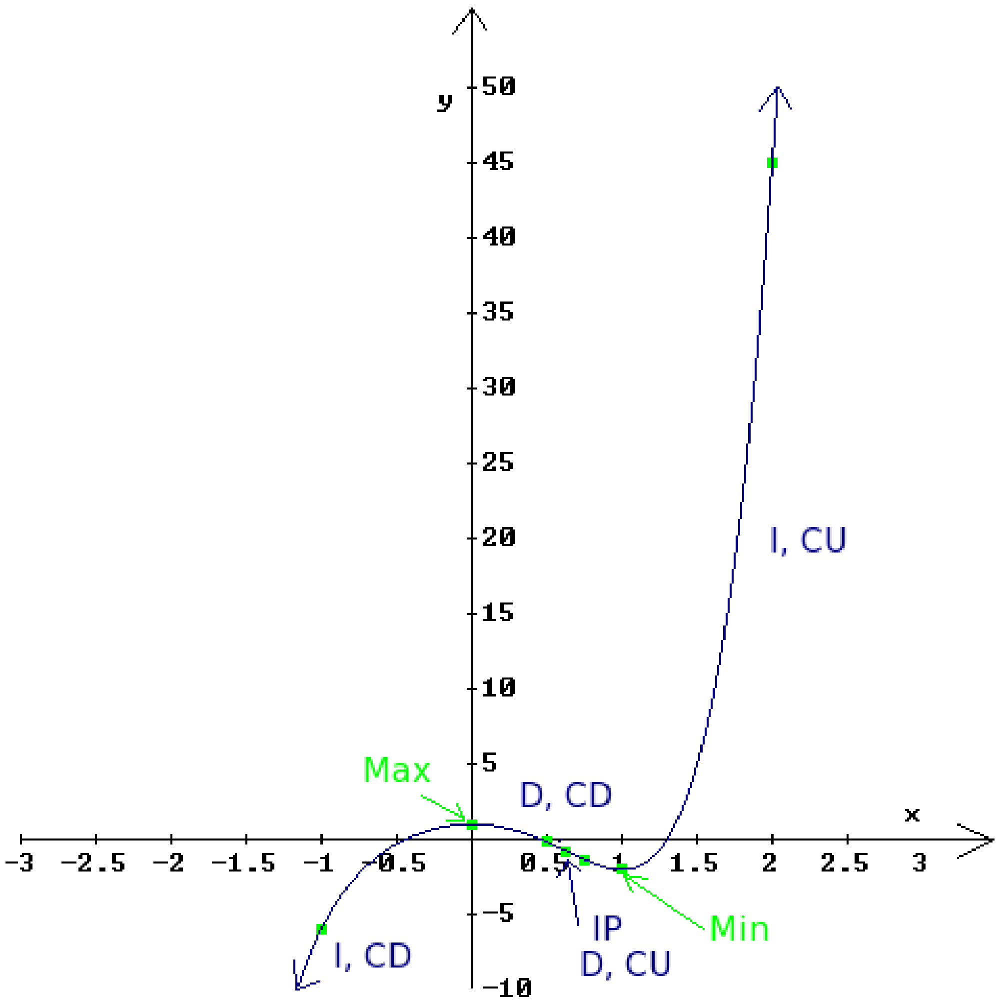
$$f\left(\frac{1}{\sqrt[3]{4}}\right) = 2\left(\frac{1}{\sqrt[3]{4}}\right)^5 - 5\left(\frac{1}{\sqrt[3]{4}}\right)^2 + 1$$

$$= 2\left(\frac{1}{4^{\frac{1}{3}}}\right)^5 - 5\left(\frac{1}{4^{\frac{1}{3}}}\right)^2 + 1$$

$$= 2\left(\frac{1}{4^{\frac{5}{3}}}\right) - 5\left(\frac{1}{4^{\frac{2}{3}}}\right) + 1$$

$$= \frac{2}{4^{\frac{5}{3}}} - \frac{5}{4^{\frac{2}{3}}} + 1 = \frac{2-5(4)+4^{\frac{5}{3}}}{4^{\frac{5}{3}}} = \frac{2-20+4^{\frac{5}{3}}}{4^{\frac{5}{3}}} = \frac{-18+4^{\frac{5}{3}}}{4^{\frac{5}{3}}} \approx -0.78583$$

8. Sketch



y
50
45
40
35
30
25
20
15
10
5
-5
-10
x
-3
-2.5
-2
-1.5
-1
-0.5
0.5
1
1.5
2
2.5
3
Max
D, CD
I, CU
I, CD
IP
D, CU
Min