Section 3.7

This section is dedicated to figuring out limits that give us answers like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ which make no mathematical sense. (indeterminate forms)

Theorem: L'Hopital's Rule (1st Form)

Suppose $f(a) = g(a) = 0$ and that $f'(a)$ and $g'(a)$ exist <u>and</u> that $g'(a) \neq 0$.

Then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

Why does this work? By definition of a derivative

$$\frac{f'(a)}{g'(a)} = \frac{\lim\limits_{x \to a} \frac{f(x)-f(x)}{x-a}}{\lim\limits_{x \to a} \frac{g(x)-g(x)}{x-a}} = \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \lim_{x \to a} \frac{f(x)-f(a)}{g(x)-g(a)} \text{ but } f(a) = g(a) = 0$$

$$= \lim_{x \to a} \frac{f(x)}{g(x)}$$

Hence $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

How might I use this?

Example: $\lim\limits_{x \to 0} \frac{3x - \sin x}{x} = \frac{3(0)-\sin(0)}{0} = \frac{0}{0}$ indeterminate

Old Way:

$$= \lim_{x \to 0} \frac{3x}{x} - \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} 3-1 = 3-1 = 2$$

Apply L'Hopital's Rule:

$$= \left. \frac{3-\cos x}{1} \right|_{x=0} = \frac{3-\cos(0)}{1} = 3-1 = 2$$

Example: $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \frac{\sqrt{1+0}-1}{0} = \frac{0}{0}$ indeterminate

Old Way:

$$= \lim_{x \to 0} \frac{\left(\sqrt{1+x}-1\right)\left(\sqrt{1+x}+1\right)}{x\left(\sqrt{1+x}+1\right)} = \lim_{x \to 0} \frac{\cancel{1}+\cancel{x}-\cancel{1}}{\cancel{x}\left(\sqrt{1+x}+1\right)} = \lim_{x \to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{1+1} = \frac{1}{2}$$

Apply L'Hopital's Rule:

$$= \left. \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}(1)-0}{1} \right|_{x=0} = \frac{1}{2} \cdot \frac{1}{\sqrt{1+0}} = \frac{1}{2}$$

Example: $\lim_{x \to 0} \frac{x-\sin x}{x^3}$

Apply L'Hopital's Rule:

$$= \left. \frac{1-\cos x}{3x^2} \right|_{x=0} = \frac{1-1}{0} = \frac{0}{0}$$ uh-oh, it is still indeterminate

What to do? Apply L'Hopital's Rule again:

$$= \left. \frac{0+\sin x}{6x} \right|_{x=0} = \frac{0}{0}$$

And, again:

$$= \left. \frac{\cos x}{6} \right|_{x=0} = \frac{1}{6}$$ wow...

L'Hopital's Rule (2nd form)

Suppose *f(a)* = *g(a)* = 0 and that *f* and *g* are differentiable in an area containing *a*. Also, suppose $g'(x) \neq 0$ if $x \neq a$.

Then: $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Example: $\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2}$

Apply L'Hopital's Rule:

$$= \frac{\sqrt{1+0} - 1 - \frac{0}{2}}{0} = \frac{0}{0}$$ Bad

Apply L'Hopital's Rule:

$$= \lim_{x \to 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}} - 0 - \frac{1}{2}}{2x} = \frac{\frac{1}{2}(1)^{-\frac{1}{2}} - \frac{1}{2}}{2(0)} = \frac{0}{0} \qquad \text{Bad}$$

Apply L'Hopital's Rule:

$$= \lim_{x \to 0} \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-\frac{3}{2}}}{2} = \frac{-\frac{1}{4}(1+0)^{-\frac{3}{2}}}{2} = \frac{-\frac{1}{4}}{2} = -\frac{1}{8} \qquad \text{OK}$$