# Section 4.3 Derivatives and Shapes of Graphs

Recall that a derivative represents an instantaneous slope. so, if you find that *f ' (x)* is positive, you would expect that *f(x)* is going uphill. Likewise if you find that *f ' (x)* is negative, you would expect that *f(x)* is going downhill. These intuitions are true.

Increasing / Decreasing Test (pg. 212)

1. If $f'(x) > 0$ on an interval, then *f* is increasing on that interval
2. If $f'(x) < 0$ on an interval, then *f* is decreasing on that interval

**Example** Find where the function $x^4 - 4x - 1$ is increasing and where it is decreasing.

First: we must find critical points.

$$f'(x) = 4x^3 - 4$$
$$f'(x) = 4(x^3 - 1)$$
$$f'(x) = 4(x-1)(x^2+x+1)$$

So, $c = \{1\}$

$$f'(c) = 0 = 4(c-1)(c^2+c+1)$$

$$c - 1 = 0 \qquad c^2 + c + 1 = 0$$
$$c = 1 \qquad c = \text{imaginary}$$

Next: We need to develop a table to help us visualize the intervals around the CP

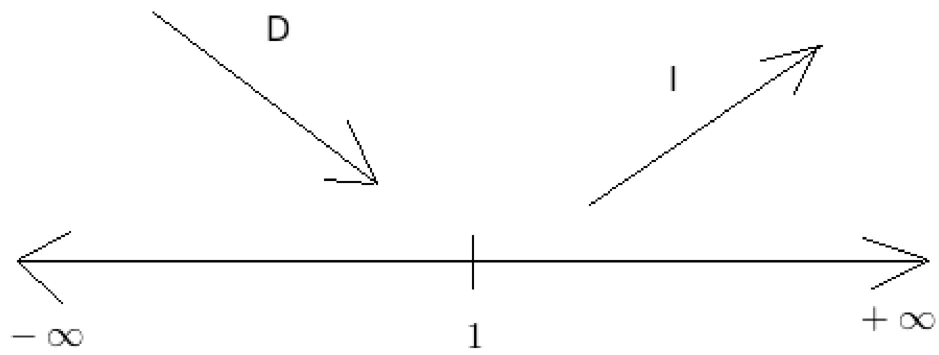
| Interval / CP | Test Point | $f'(x) = 4(x-1)(x^2+x+1)$ | Inc/Dec |
|---|---|---|---|
| $(-\infty\ ,\ 1)$ | 0 | (+)(-)(+) = (-) | Decreasing |
| 1 | 1 | (+)(0)(+) = 0 | |
| $(1\ ,\ \infty)$ | 2 | (+)(+)(+) = (+) | Increasing |

Then: State what you have found:

On $x \in (-\infty\ ,\ 1)$, *f(x)* is decreasing

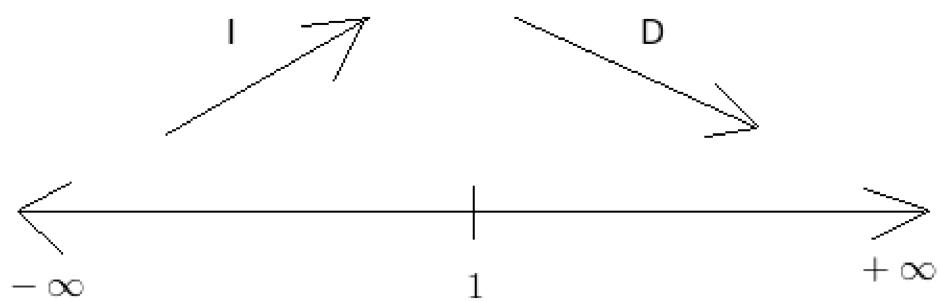
On $x \in (1\ ,\ \infty)$, *f(x)* is increasing

Visualize what this means...




What is happening at $x = 1$? Local Minimum

Likewise if you had seen:




Local Maximum

**The First Derivative Test**

Suppose that $c$ is a critical number of a continuous function $f$.

a. If $f'$ changes from (+) to (-) at $c$, then $f$ has a local maximum at $c$.
b. If $f'$ changes from (-) to (+) at $c$, then $f$ has a local minimum at $c$.
c. If $f'$ does not change sign at $c$ (so (+) to (+) or (-) to (-)), then $f$ has no local maximum or local minimum at $c$.

**Example** Find the local maximum and minimum of: $f(x) = x - 2\sin x$ when $0 < x < 3\pi$

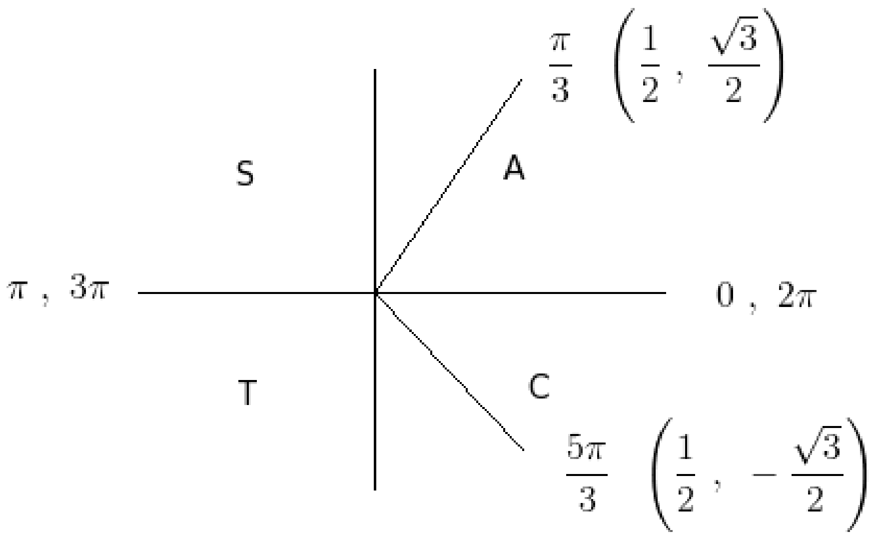
First: critical points

$$f'(x) = 1 - 2\cos x$$

$$f'(c) = 0 = 1 - 2\cos c$$

$$-1 = -2\cos c$$
$$\frac{1}{2} = \cos c$$

Where does this happen?  (Between 0 and 3π)




Unit Circle

$$c = \left\{\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}\right\}$$

Next: We need to develop a table

| Int. / CP | Test Pt | Sign of $f(x) = 1 - \cos x$ | Inc / Dec? |
| --- | --- | --- | --- |
| $\left(0\ ,\ \frac{\pi}{3}\right)$ | $\frac{\pi}{4}$ | $1-2\left(\frac{\sqrt{2}}{2}\right) = 1-\sqrt{2} = (-)$ | Decreasing |
| $\frac{\pi}{3}$ | $\frac{\pi}{3}$ | $1-2\left(\frac{1}{2}\right) = 0$ | |
| $\left(\frac{\pi}{3}\ ,\ \frac{5\pi}{3}\right)$ | $\pi$ | $1-2(-1) = 1+2 = (+)$ | Increasing |
| $\frac{5\pi}{3}$ | $\frac{5\pi}{3}$ | $1-2\left(\frac{1}{2}\right) = 0$ | |
| $\left(\frac{5\pi}{3}\ ,\ \frac{7\pi}{3}\right)$ | $2\pi$ | $1-2(1) = 1-2 = (-)$ | Decreasing |
| $\frac{7\pi}{3}$ | $\frac{7\pi}{3}$ | $1-2\left(\frac{1}{2}\right) = 0$ | |
| $\left(\frac{7\pi}{3}\ ,\ 3\pi\right)$ | $\frac{5\pi}{2}$ | $1-2(0) = 1$ | Increasing |

Then: State what you have found.

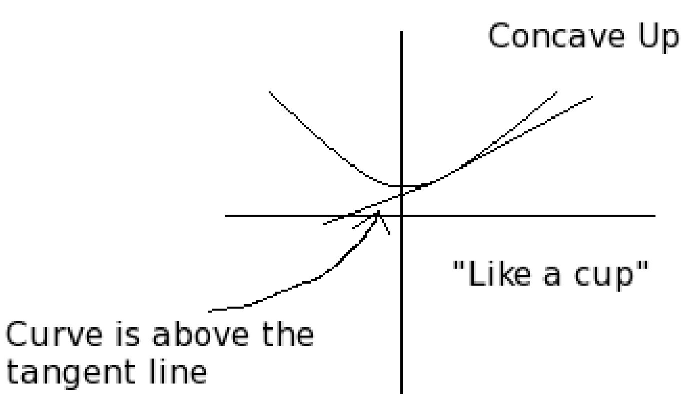
On $x \in \left(0\ ,\ \frac{\pi}{3}\right) \cup \left(\frac{5\pi}{3}\ ,\ \frac{7\pi}{3}\right)$, $f(x)$ is decreasing

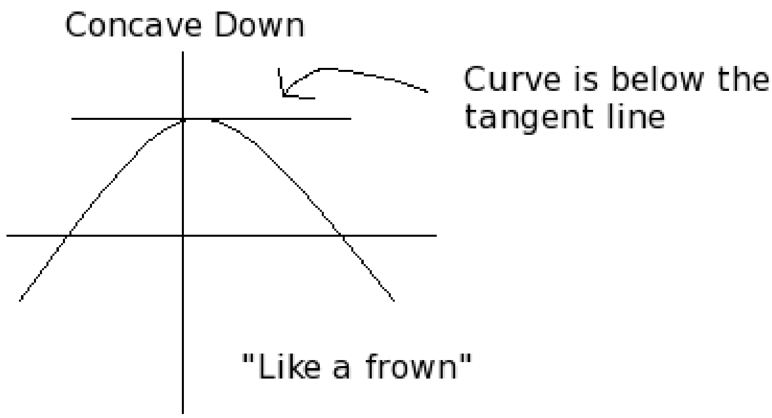
On $x \in \left(\frac{\pi}{3}\ ,\ \frac{5\pi}{3}\right) \cup \left(\frac{7\pi}{3}\ ,\ 3\pi\right)$, $f(x)$ is increasing

So, what about $f''$? $f'$ tells us increasing and decreasing – surely $f''$ must mean something.

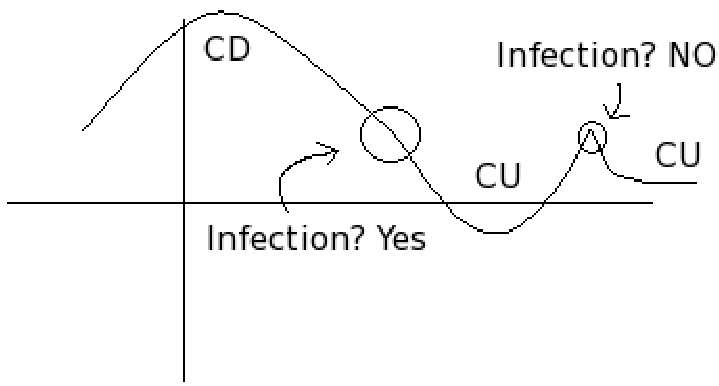
$f''$ acts like the slope of $f'$ (just like $f'$ acts like the slope of $f$). So, if I know that the slope of my slope is doing something – what does that mean for $f$?

It deals with concavity.






**Definition**: A point *P* on a curve $y = f(x)$ is called an <u>inflection</u> <u>point</u> if *f* is continuous there and a curve changes from concave up to concave down OR concave down to concave up.




So, $f''$ changes from (+) to (-) OR $f''$ changes from (-) to (+)

What happens in between? $f'' = 0$

And that is where a "possible" inflection point may be.

**Concavity Test**

a. If $f'' > 0$ for all *x* in *I*, then the graph of *f* is concave up on *I*.
b. If $f'' < 0$ for all *x* in *I*, then the graph of *f* is concave down on *I*.

**The Second Derivative Test**

Suppose that $f''$ is continuous near *c*.

a. If $f'(c) = 0$ and $f''(c) > 0$ *f* has a local minimum at *c*.
b. If $f'(c) = 0$ and $f''(c) < 0$ *f* has a local maximum at *c*.

**Example** pg. 21B #29) $A(x) = x\sqrt{x+3} = x(x+3)^{\frac{1}{2}}$

a. Find intervals of increasing and decreasing
b. Find local maximum(s) and minimum(s)
c. Find intervals of concavity and the inflection points
d. Sketch the graph of *A*.

1st: Find $A'(x)$

$$A'(x) = x\left(\frac{1}{2}\right)(x+3)^{-\frac{1}{2}}(1) + (x+3)^{\frac{1}{2}}(1)$$

$$= \frac{x}{2(x+3)^{\frac{1}{2}}} + (x+3)^{\frac{1}{2}}$$

$$= \frac{x+2(x+3)}{2(x+3)^{\frac{1}{2}}}$$

$$= \frac{3x+6}{2(x+3)^{\frac{1}{2}}}$$

$$= \frac{3(x+2)}{2(x+3)^{\frac{1}{2}}}$$

2nd: Damian of $A(x)$ : $x \geq -3 \quad \Rightarrow \quad x \in [-3\ ,\ \infty)$

3rd: Critical Points

$A'(x) =$ OR $A'(x) =$ DNE

$x+2 = 0$ $\qquad$ $(x+3)^{\frac{1}{2}} = 0$

$x = -2$ $\qquad$ $x = -3$

Which of these are in our domain? Both: $c = \{-3\ ,\ -2\}$

4th: Find $A''(x)$

$$A''(x) = \frac{2(x+3)^{\frac{1}{2}} - 3(x+2)\cancel{2}\left(\frac{1}{\cancel{2}}\right)(x+3)^{-\frac{1}{2}}}{\left(2(x+3)^{\frac{1}{2}}\right)^2}$$

$$= \frac{6(x+3)^{\frac{1}{2}} - \frac{3(x+2)}{(x+3)^{\frac{1}{2}}}}{4(x+3)}$$

$$= \frac{6(x+3)^{\frac{1}{2}}}{4(x+3)} - \frac{3(x+2)}{4(x+3)(x+3)^{\frac{1}{2}}}$$

$$= \frac{6}{4(x+3)^{\frac{1}{2}}} - \frac{3(x+2)}{4(x+3)^{\frac{3}{2}}}$$

Algebra!

$$= \frac{6(x+3) - 3(x+2)}{4(x+3)^{\frac{3}{2}}}$$

$$= \frac{6x + 18 - 3x - 6}{4(x+3)^{\frac{3}{2}}}$$

$$= \frac{3x + 12}{4(x+3)^{\frac{3}{2}}}$$

$$= \frac{3(x+4)}{4(x+3)^{\frac{3}{2}}}$$

5th: Possible Inflection Points $A''(x) = 0$

$$3(x+4) = 0$$
$$x = -4$$

But this it outside of our domain, so no Inflection Points.

6th: Set Up a Table

| Interval CP / IP | Test Point | $A'(x) = \frac{3(x+2)}{2(x+3)^{\frac{1}{2}}}$ | Increasing Decreasing | $A''(x) = \frac{3(x+4)}{4(x+3)^{\frac{3}{2}}}$ | CU CD Max Min | $A(x) = x\sqrt{x+3}$ |
|---|---|---|---|---|---|---|
| -3 | -3 | $\frac{(+)(-)}{(+)(0)} = \text{DNE}$ | | $\frac{(+)(+)}{(+)(0)} = \text{DNE}$ | | $(-3)(0) = 0$ |
| $(-3\ ,\ -2)$ | -2.5 | $\frac{(+)(-)}{(+)(+)} = (-)$ | Decreasing | $\frac{(+)(+)}{(+)(+)} = (+)$ | CU | $\left(-\frac{5}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = -\frac{5}{2\sqrt{2}}$ |
| -2 | -2 | $\frac{(+)(0)}{(+)(+)} = 0$ | | $\frac{(+)(+)}{(+)(+)} = (+)$ | Local Min | $(-2)(1) = -2$ |
| $(-2\ ,\ \infty)$ | 0 | $\frac{(+)(+)}{(+)(+)} = (+)$ | Increasing | $\frac{(+)(+)}{(+)(+)} = (+)$ | CU | $(0)\sqrt{3} = 0$ |

7th: Graph from your table

1. Label Convenient points
2. Follow I/D/CU/CD/Max and Min