# Section 4.1 Max and Min (Optimization Problems)

**Definition** Absolute maximum - "Biggest Value"
*f* has an absolute maximum if $f(c) \geq f(x) \quad \forall x \in D$

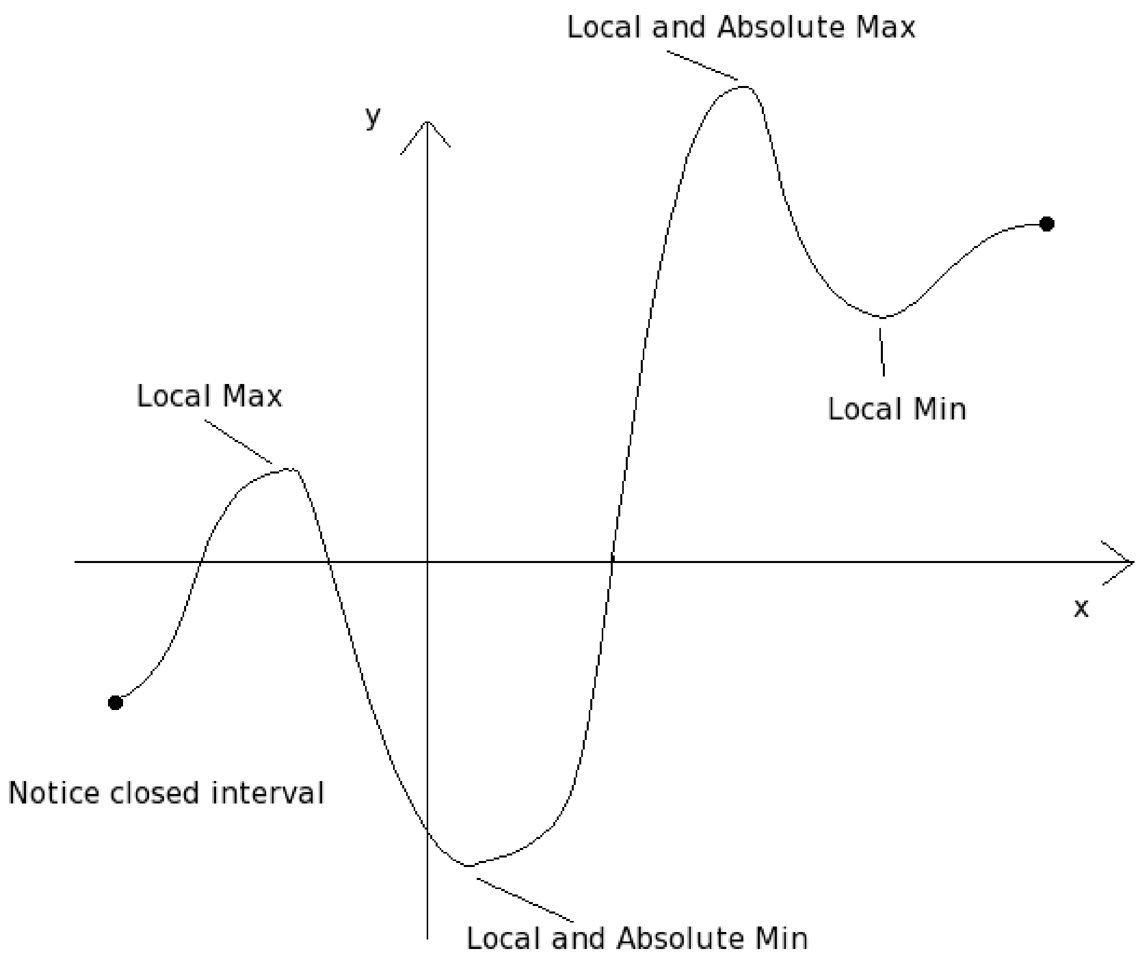
**Definition** Absolute minimum - "Smallest Value"
*f* has an absolute minimum if $f(c) \leq f(x) \quad \forall x \in D$

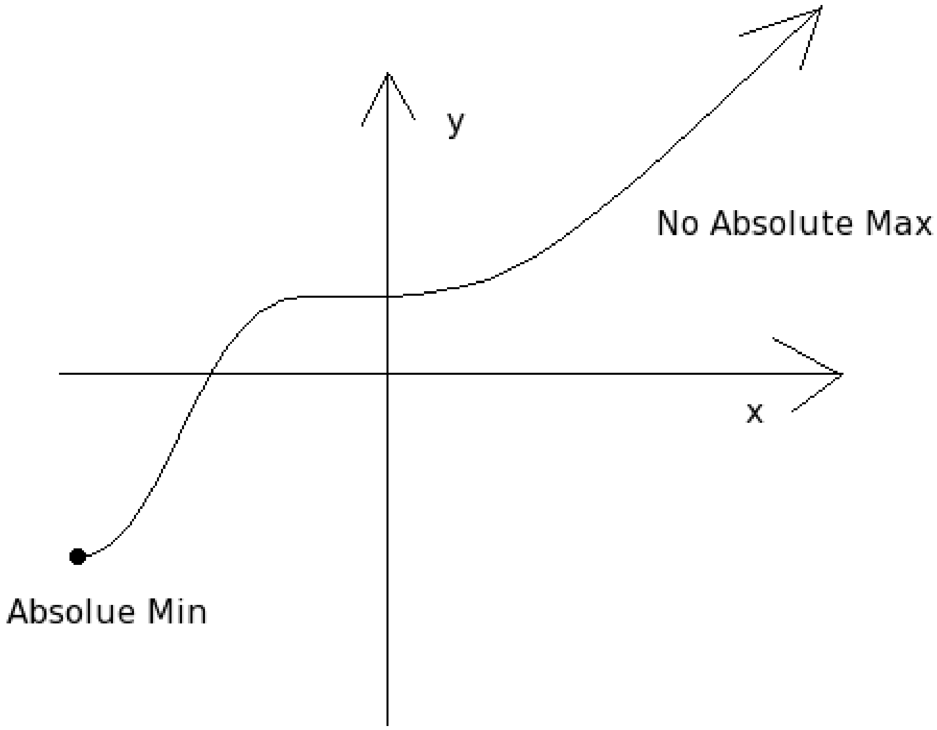
NOTE: Both are called <u>Extrema</u>

**Definition** Local maximum - "Biggest" on an interval
*f* has a local maximum at *c* if $f(c) \geq f(x)$ when *x* is near *c*.

**Definition** Local minimum - "Smallest" on an interval
*f* has a local minimum at *c* if $f(c) \leq f(x)$ when *x* is near *c*.

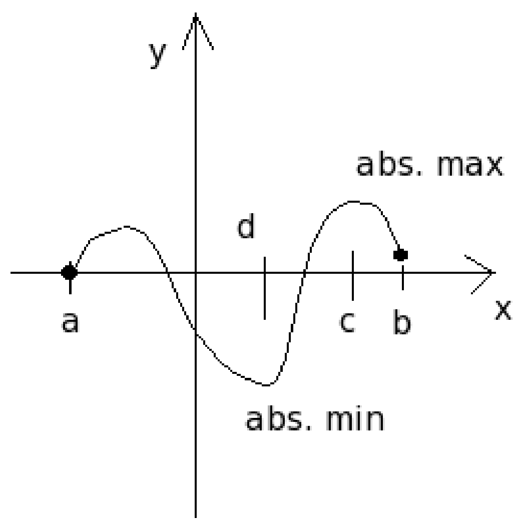
Examples:

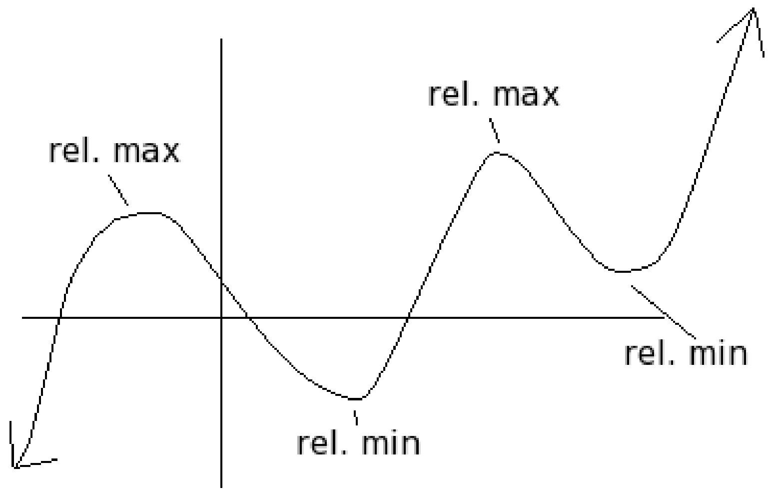





Open or partially close interval

**Theorem** If $f$ is continuous on a close interval $[a\ ,\ b]$, then $f$ attains an absolute maximum value $f\,(c)$ and an absolute minimum value $f\,(d)$ at some numbers $c$ and $d$ in $[a\ ,\ b]$.






No Abs. Max or Min

What do you notice about where the relative maximum and minimum occur?

- Hump or a valley
- Particular point where slope changes from + to – or – to +
- Local tangent line is horizontal

KEY: All of these things indicate that at *c* (where the local max or min occurs) that $f'(c) = 0$

This is called Fermat's Theorem:
If *f* has a local maximum or minimum at *c*, and if $f'(c)$ exists, then $f'(c) = 0$

In fact, we give these *c* a special name.

**Definition** A critical number of a function *f* is a number *c* in the domain of *f* such that either $f'(c) = 0$ or $f'(c)$ does not exist.

**Example** 25) Find the critical numbers of $f(x) = x^3 + 3x^2 - 24x$

1. Find derivative and simplify

   $f'(x) = 3x^2 + 6x - 24$

2. Set $f'(x) = 0$

   $3(x+4)(x-2) = 0$

3. Solve

   $x + 4 = 0$ $\qquad$ $x - 2 = 0$
   $x = -4$ $\qquad$ $x = 2$

Critical Numbers: $\{-4\ ,\ 2\}$

**Example** 43) Find the absolute maximum and minimum on $[-1\ ,\ 2]$ for $f(t) = t\sqrt{4-t^2}$.

1. Find derivative and simplify

$$f'(t) = t\left(\frac{1}{\not{2}}\right)\left(4-t^2\right)^{-\frac{1}{2}}(-\not{2}t)+\left(4-t^2\right)^{\frac{1}{2}}(1)$$

$$= -\frac{t^2}{\left(4-t^2\right)^{\frac{1}{2}}}+\left(4-t^2\right)^{\frac{1}{2}}$$

$$= \frac{-t^2+4-t^2}{\left(4-t^2\right)^{\frac{1}{2}}}$$

$$f'(t) = \frac{4-2t^2}{\left(4-t^2\right)^{\frac{1}{2}}}$$

2. Find critical points

Set $f'(t) = 0$ $\Rightarrow$

$$\begin{aligned} 4-t^2 &= 0 \\ -2t^2 &= -4 \\ t^2 &= 2 \\ t &= \pm\sqrt{2} \end{aligned}$$

numerator = 0

Set $f'(t) = \text{DNE}$ $\Rightarrow$

$$\begin{aligned} \left(4-t^2\right)^{\frac{1}{2}} &= 0 \\ 4-t^2 &= 0 \\ t^2 &= 4 \\ t &= \pm 2 \end{aligned}$$

denominator = 0

3. Which critical points are inside our interval? $[-1\ ,\ 2]$

   Only $c = \{-\sqrt{2}\ ,\ +\sqrt{2}\ ,\ 2\}$

4. Check $f(c)$ at all critical points and endpoints.

| Points | $f(t) = t\sqrt{4-t^2}$ |
|---|---|
| $-1$ | $(-1)(4-1)^{\frac{1}{2}} = -\sqrt{3}$ |
| $-\sqrt{2}$ | $(-\sqrt{2})(4-2)^{\frac{1}{2}} = -\sqrt{2}\sqrt{2} = -2 \leftarrow$ smallest value |
| $\sqrt{2}$ | $(\sqrt{2})(4-2)^{\frac{1}{2}} = \sqrt{2}\sqrt{2} = 2 \leftarrow$ largest value |
| $2$ | $(2)(4-4)^{\frac{1}{2}} = 0$ |

5. Abs maximum at $x = \sqrt{2}$ , $f(\sqrt{2}) = 2$
   Abs minimum at $x = -\sqrt{2}$ , $f(-\sqrt{2}) = -2$

**Example** 55) Between 0 °C and 30 °C, the volume $V$ (in $cm^3$) of 1 kg of $H_20$ at a temperature $T$ is given by:

$$V(T) = 999.87 - 0.06426\ T + 0.0085043\ T^2 - 0.0000679\ T^3$$

*NOTE, it may be easier to allow the form to be

$$V(T) = a - bT + cT^2 - dT^3$$

where $a = 999.87$ , $b = 0.06426$ , $c = 0.0085043$ , $d = 0.0000679$
and substitute the values for *a*, *b*, *c*, and *d* in the end.

Find $T$ with maximum density. (i.e. use smallest Volume)

$$\rho = \text{density} = \frac{\text{mass}}{\text{volume}} = \frac{\text{g}}{\text{cm}^3}$$ to make $\rho$ big, $V$ must be small.

$$V'(T) = -0.06426 + 2(0.0085043)T - 3(0.0000679)T^2$$
$$V'(T) = -0.06426 + 0.01700860T - 0.00020370T^2$$

$V'(T) = 0$ Use Quadratic Formula

$$T = \frac{-0.01700860 \pm \sqrt{(0.01700860)^2 - 4(-0.00020370)(-0.06426)}}{2(-0.00020370)}$$

$$T = \frac{-0.01700860 \pm \sqrt{0.00028929 - 0.00005236}}{-0.00052740}$$

$$T = \frac{-0.01700860 \pm 0.01539253}{-0.00052740}$$

$T$ = 3.9665 °C or 79.5318 °C (Note: 79.5318 °C is outside given interval)

$$\rho(V) = \frac{1\ \text{kg}}{V} = \frac{1000\,\text{g}}{V}$$

$$\rho(0) = \frac{1000}{V(0)} = 1.00013\ \frac{\text{g}}{\text{cm}^3}$$

$$\rho(3.9665) = \frac{1000}{V(3.9665)} = 1.00026\ \frac{\text{g}}{\text{cm}^3}$$

$$\rho(30) = \frac{1000}{V(30)} = 0.99625\ \frac{\text{g}}{\text{cm}^3}$$

T = 3.9665 °C is the largest.