# Section 3.3 Derivatives of Logarithmic and Exponential Functions

**Theorem:** If $f(x) = \log_a x$

Then $f'(x) = \frac{1}{x}\log_a e$

Change of Base Rule: $\log_a e = \frac{\log_e e}{\log_e a} = \frac{1}{\ln a}$

So $\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x}\cdot\frac{1}{\ln a}$

Likewise

$$\frac{d}{dx}\left(\log_e x\right) = \frac{d}{dx}(\ln x) = \frac{1}{x}\cdot\frac{1}{\ln e} = \frac{1}{x}(1) = \frac{1}{x}$$

**Example:** Find *f'(x)*:

$$f(x) = \log_2(1-3x)$$

Apply Chain Rule:

$$\begin{aligned} f'(x) &= \frac{1}{1-3x}\cdot\frac{1}{\ln 2}\cdot\frac{d}{dx}(1-3x) \\ &= \frac{1}{1-3x}\cdot\frac{1}{\ln 2}\cdot(-3) \\ &= -\frac{3}{(1-3x)\ln 2} \end{aligned}$$

**Example:** Find *f'(x)*:

$$f(x) = \ln\sqrt[5]{x} = \ln\left[x^{\frac{1}{5}}\right]$$

Apply Chain Rule:

$$\begin{aligned} f'(x) &= \frac{1}{x^{\frac{1}{5}}}\cdot\frac{1}{\ln e}\cdot\frac{d}{dx}\left(x^{\frac{1}{5}}\right) \\ &= \frac{1}{x^{\frac{1}{5}}}\cdot\frac{1}{\ln e}\cdot\frac{1}{5}x^{-\frac{4}{5}} \end{aligned}$$

$$= \frac{1}{x^{\frac{1}{5}}} \cdot \frac{1}{5} \cdot \frac{1}{x^{\frac{4}{5}}} = \frac{1}{5x}$$

## Derivative of Exponential ($a$ > 0)

$$\frac{d}{dx}\left(a^x\right) = a^x \ln a$$

Likewise: $\frac{d}{dx}\left(e^x\right) = e^x \ln e = e^x(1) = e^x$

**Example:** Find *f'(x)*:

$$f(x) = \frac{e^x}{1+x}$$

$$f'(x) = \frac{(1+x)\frac{d}{dx}\left(e^x\right) - \left(e^x\right)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{(1+x)\left(e^x\right) - \left(e^x\right)(1)}{(1+x)^2}$$

$$= \frac{e^x + x e^x - e^x}{(1+x)^2}$$

$$= \frac{x e^x}{(1+x)^2}$$

**Example:** Find *f′(x)*:

$$f(x) = \log_5\left(x \cdot e^x\right)$$

$$f'(x) = \frac{1}{\underbrace{(\quad)}_{\text{Fill in}}} \cdot \frac{1}{\ln 5} \cdot \frac{d}{dx}\underbrace{(\quad)}_{\text{Fill in}}$$

$$= \frac{1}{x\,e^x} \cdot \frac{1}{\ln 5} \cdot \frac{d}{dx}\left(x\,e^x\right)$$

$$= \frac{1}{x\,e^x} \cdot \frac{1}{\ln 5} \cdot \left[x\,\frac{d}{dx}\left(e^x\right) + e^x\,\frac{d}{dx}(x)\right] \quad \text{product rule}$$

$$= \frac{1}{x\,e^x} \cdot \frac{1}{\ln 5} \cdot \left(x\,e^x + e^x\right)$$

$$= \frac{1}{x\,\cancel{e^x}} \cdot \frac{1}{\ln 5} \cdot \cancel{e^x}\,(x+1)$$

$$= \frac{x+1}{x \ln 5}$$

**Example:** Find *y′*:

$$y = e^{-5x} \cos 3x$$

$$y' = e^{-5x}\,\frac{d}{dx}(\cos 3x) + (\cos 3x)\,\frac{d}{dx}\left(e^{-5x}\right)$$

$$= \underbrace{e^{-5x}\left(-\sin(3x)\right)\frac{d}{dx}(3x) + \cos(3x)\,e^{-5x}\,\frac{d}{dx}(-5x)}_{\text{Product Rule}}$$

$$= e^{-5x}\left(-\sin(3x)\right)(3) + \cos(3x)\,e^{-5x}(-5)$$

$$= -3\,e^{-5x}\sin(3x) - 5\,e^{-5x}\cos(3x)$$

**Example:** Find *y′*:

$$y = \left[\ln\left(1+e^x\right)\right]^2$$

$$y' = \underbrace{2\left[\ln\left(1+e^x\right)\right]^1 \frac{d}{dx}\left[\ln\left(1+e^x\right)\right]}_{\text{Chain Rule}}$$

$$= 2\ln\left(1+e^x\right)\underbrace{\left(\frac{1}{1+e^x}\right)\frac{d}{dx}\left(1+e^x\right)}_{\text{Chain Rule Again}}$$

$$= 2\ln\left(1+e^x\right)\left(\frac{1}{1+e^x}\right)e^x$$

$$= \frac{2\,e^x \ln\left(1+e^x\right)}{1+e^x}$$

# Derivative of a Logarithm

Sometimes logarithms can make taking a derivative easier because we can use their super powers to break functions apart.

We are going to use:

$$\log_a(xy) = \log_a x + \log_a y$$
$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a x^r = r\cdot\log_a x$$

to our advantage.

**Example:** Take the derivative of $y = \sqrt{x}\,e^{x^2}\left(x^2+1\right)^{10}$ using logarithms.

At 1st glance, this function is a mess of product and chain rules – the old way – but watch...

1. Take the logarithms of both sides and simplify as much as you like:

$$\ln y = \ln\left[\sqrt{x}\,e^{x^2}\left(x^2+1\right)^{10}\right]$$

Simplify Lots:

$$\ln y = \ln\sqrt{x} + \ln\left(e^{x^2}\right) + \ln\left(x^2+1\right)^{10}$$
$$\ln y = \ln\left(x^{\frac{1}{2}}\right) + x^2\ln e + 10\ln\left(x^2+1\right)$$
$$\ln y = \frac{1}{2}\ln(x) + x^2(1) + 10\ln\left(x^2+1\right)$$

2. Implicitly Differentiate:

$$\frac{1}{y}\cdot\frac{1}{\ln e}\cdot\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x}\cdot\frac{1}{\ln e}\right] + 2\mathrm{x} + 10\left[\frac{1}{x^2+1}\cdot\frac{1}{\ln e}\frac{d}{dx}\left(x^2+1\right)\right]$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2\mathrm{x}} + 2\mathrm{x} + 10\left[\frac{1}{x^2+1}\cdot 2\mathrm{x}\right]$$
$$= \frac{1}{y}\frac{dy}{dx} = \frac{1}{2\mathrm{x}} + 2\mathrm{x} + \frac{20\mathrm{x}}{x^2+1}$$

3. Isolate $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y\left(\frac{1}{2\mathrm{x}} + 2\mathrm{x} + \frac{20\mathrm{x}}{x^2+1}\right)$$

4. Substitute in value for *y* from the original form:

$$\frac{dy}{dx} = \left(\sqrt{x}\,e^{x^2}\left(x^2+1\right)^{10}\right)\left(\frac{1}{2\mathrm{x}} + 2\mathrm{x} + \frac{20\mathrm{x}}{x^2+1}\right)$$

**Example:** Take the derivative of $y = \sqrt[4]{\dfrac{x^2+1}{x^2-1}}$ using logarithms.

Once gain, at 1st glance, this function is a mess of product and chain rules – the old way – but watch...

1. Take the logarithms of both sides and simplify as much as you like:

$$\ln y = \ln\left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{4}}$$
$$= \frac{1}{4}\ln\left(\frac{x^2+1}{x^2-1}\right)$$
$$= \frac{1}{4}\left[\ln\left(x^2+1\right) - \ln\left(x^2-1\right)\right]$$

2. Implicitly Differentiate:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{4}\left[\frac{1}{x^2+1}\frac{d}{dx}\left(x^2+1\right) - \frac{1}{x^2-1}\frac{d}{dx}\left(x^2-1\right)\right]$$
$$= \frac{1}{4}\left[\frac{1}{x^2+1}(2\text{x}) - \frac{1}{x^2-1}(2\text{x})\right]$$

3. Isolate $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{y}{4}\left[\frac{2\text{x}}{x^2+1} - \frac{2\text{x}}{x^2-1}\right]$$

4. Substitute in value for *y* from the original form:

$$\frac{dy}{dx} = \sqrt[4]{\frac{x^2+1}{x^2-1}}\cdot\frac{1}{4}\cdot\left[\frac{2\text{x}}{x^2+1} - \frac{2\text{x}}{x^2-1}\right]$$

Reduce further if you wish.

**Example:** Find $y'$ if $y = (\sin x)^{\ln x}$

*There is no way to handle this without logarithms because the exponent is not a real number.

1. Take the logarithms of both sides and simplify as much as you like:

$$\ln y = \ln(\sin x)^{\ln x}$$
$$= (\ln x)\cdot\ln(\sin x)$$

2. Implicitly Differentiate:

$$\frac{1}{y}\frac{dy}{dx} = \ln x \frac{1}{\sin x}\frac{d}{dx}(\sin x) + \ln(\sin x)\frac{1}{x}$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{\ln x}{\sin x}(\cos x) + \frac{\ln(\sin x)}{x}$$

3. Isolate $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y\left(\cot x \ \ln x + \frac{\ln(\sin x)}{x}\right)$$

4. Substitute in value for $y$ from the original form:

$$\frac{dy}{dx} = (\sin x)^{\ln x}\left(\cot x \ \ln x + \frac{\ln(\sin x)}{x}\right)$$

**Example:** Find $y'$ if $y = x^{\cos x}$

$$\ln y = \ln x^{\cos x}$$
$$\ln y = \cos x \ln x$$
$$\underbrace{\frac{1}{y}\frac{dy}{dx} = \cos \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(\cos x)}_{\text{Implice \& Product Rules}}$$
$$\frac{1}{y}\frac{dy}{dx} = \cos x \cdot \frac{1}{x} + \ln x(-\sin x)$$
$$\frac{dy}{dx} = y\left[\frac{\cos x}{x} - (\ln x)(\sin x)\right]$$
$$\frac{dy}{dx} = \ln x^{\cos x}\left[\frac{\cos x}{x} - (\ln x)(\sin x)\right]$$

**Example:** Find $y'$ if $y = (\sqrt{x})^x$

$$y = \left(x^{\frac{1}{2}}\right)^x = x^{\frac{x}{2}}$$
$$\ln y = \ln\left(x^{\frac{x}{2}}\right)$$
$$\ln y = \frac{x}{2}\ln x$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{2}\cdot\frac{1}{x} + \ln(x)\left(\frac{1}{2}\right)$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2} + \frac{\ln x}{2}$$

$$\frac{dy}{dx} = y\left[\frac{1}{2} + \frac{\ln x}{2}\right]$$

$$\frac{dy}{dx} = \left(\sqrt{x}\right)^x\left[\frac{1}{2} + \frac{\ln x}{2}\right]$$