Section 2.6 Implicit Differentiation

How do we handle relationships between *x* and *y* that we cannot completely separate?

Example: We can handle $y = \sqrt{x^3+1}$ because all the *x's* and *y's* are separate.

$y = f(x)$ And chain...

$$y = (x^3+1)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(x^3+1)^{-\frac{1}{2}}\frac{d}{dx}(x^3+1)$$
$$= \frac{1}{2}(x^3+1)^{-\frac{1}{2}}(3x^2)$$

But, what do we do with $x^2 + y^2 = 25$?

Sometimes we can explicitly solve:
$$y = \pm\sqrt{25-x^2} \Rightarrow f(x) = \sqrt{25-x^2} \cup g(x) = -\sqrt{25-x^2}$$

But what do we do about $x^3 + y^3 = 6xy$ which cannot be solved explicitly?

Example: Implicit. Find $\frac{dy}{dx}$ implicitly.

$$x^2+y^2 = 25$$
$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25)$$
$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$
$$2x+\underbrace{2y\frac{d}{dx}(y)}_{\text{chain rule}} = 0$$

The chain rule works because *y = f(x)*.

So:
$$2x+2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2x$$

and

$$\frac{dy}{dx} = \frac{-2x}{2y}$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

Example: Find y' if $y^3+x^3=6xy$

$$\frac{d}{dx}(y^3)+\frac{d}{dx}(x^3)=\frac{d}{dx}(6xy)$$

$$\underbrace{3y^2\frac{d}{dx}(y)}_{\text{Chain Rule}}+3x^2=\underbrace{6x\frac{d}{dx}(y)+y\frac{d}{dx}(6x)}_{\text{Product Rule}}$$

$$3y^2\frac{dy}{dx}+3x^2=6x\frac{dy}{dx}+y(6)$$

Now Algebra

$$3y^2\frac{dy}{dx}-6x\frac{dy}{dx}=6y-3x^2$$

$$\frac{dy}{dx}(3y^2-6x)=6y-3x^2$$

$$\frac{dy}{dx}=\frac{6y-3x^2}{3y^2-6x}$$

$$\frac{dy}{dx}=\frac{\cancel{3}(2y-x^2)}{\cancel{3}(y^2-2x)}$$

$$\frac{dy}{dx}=\mathbf{\frac{2y-x^2}{y^2-2x}}$$

What about tangent lines?

Slope = $\frac{dy}{dx}$, but we need (x ,y) to find it.

Recall: $y^3+x^3=6xy$

Notice x = 3, y = 3

$$(3)^3+(3)^3=6(3)(3)$$
$$27+27=6(9)$$
$$54=54$$

So the point (3, 3) is on our graph.

So, m: $\frac{dy}{dx}=\frac{2(3)-(3)^2}{(3)^2-2(3)}=\frac{6-9}{9-6}=\frac{-3}{3}=-1$

And tangent line:

$$y-3=-1(x-3)$$
$$y=-x+3+3$$
$$y=-x+6$$

Example: Find $\frac{dy}{dx}$ for each

1. $x^2 - 2xy + y^3 = c$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(2xy) + \frac{d}{dx}(y^3) = \frac{d}{dx}(c)$$

$$2x - \left[2x\frac{d}{dx}(y) + (y)\frac{d}{dx}(2x)\right] - 3y^2\frac{d}{dx}(y) = 0$$

$$2x - \left[2x\frac{dy}{dx} + y(2)\right] - 3y^2\frac{dy}{dx} = 0$$

$$2x - 2x\frac{dy}{dx} - 2y - 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(-2x - 3y^2) = -2x + 2y$$

$$\frac{dy}{dx} = \mathbf{\frac{-2x+2y}{-2x-3y^2}}$$

2. $1 + x = \sin(xy^2)$

$$\frac{d}{dx}(1) + \frac{d}{dx}(x) = \frac{d}{dx}\left(\sin(xy^2)\right)$$

$$0 + 1 = \underbrace{\cos(xy^2)\frac{d}{dx}(xy^2)}_{\text{Chain Rule}}$$

$$1 = \cos(xy^2)\underbrace{\left[x\frac{d}{dx}(y^2) + (y^2)\frac{d}{dx}(x)\right]}_{\text{Product Rule}}$$

$$1 = \cos(xy^2)\left[x\underbrace{2y\frac{dy}{dx}}_{\text{chain}} + y^2(1)\right]$$

$$1 = \cos(xy^2)\left(2xy\frac{dy}{dx} + y^2\right)$$

$$1 = 2xy\cos(xy^2)\frac{dy}{dx} + y^2\cos(xy^2)$$

Isolate:

$$1 - y^2\cos(xy^2) = 2xy\cos(xy^2)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1 - y^2\cos(xy^2)}{2xy\cos(xy^2)}$$

Try on your own:

1. $y^5 + x^2y^3 = 1 + x^4y$ $\qquad \frac{dy}{dx} = \frac{4x^3y - 2xy^3}{5y^4 + 3x^2y^2 + x^4}$

2. $y \sin(x^2) = x \sin(y^2)$ $\quad \frac{dy}{dx} = \frac{\sin(y^2) - 2xy\cos(x^2)}{\sin(x^2) - 2xy\cos(y^2)}$

3. $\sqrt{x+y} = 1 + x^2 y^2$ $\quad \frac{dy}{dx} = \frac{4xy^2\sqrt{x+y} - 1}{1 - 4yx^2\sqrt{x+1}}$

4. $\sin x + \cos y = \sin x \cos y$ $\quad \frac{dy}{dx} = \frac{\cos x(\cos y - 1)}{\sin y(\sin x - 1)}$