Section 2.5 The Chain Rule

Sometimes, the rules we have learned so far are "not enough" to find a derivative, but they are parts of a larger form. When this happens, we have to CHAIN RULE to get to our derivatives.

Chain Rule: f and g are differentiable and $F = f \circ g$ is a composite function.
$F(x) = f(g(x))$, then F is also differentiable
$F'(x) = f'(g(x))g'(x)$

Examples:

1. $f(x) = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$

   We know how to handle $(...)^{\frac{1}{2}}$ and we know how to handle x^2+1
   Together we chain:

   $f'(x) = \frac{1}{2}(...)^{-\frac{1}{2}}\frac{d}{dx}(...)$ use parentheses to show the bones of the problem, then fill in.

   $f'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}\frac{d}{dx}(x^2+1)$ then, simplify

   $$f'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2\text{x}) = \frac{x}{(x^2+1)^{\frac{1}{2}}} = \frac{x}{\sqrt{x^2+1}}$$

2. $f(x) = \tan(\sin x)$

   $$f'(x) = \sec^2(\sin x)\frac{d}{dx}(\sin x) = \sec^2(\sin x)(\cos x) = \cos x \cdot \sec^2(\sin x)$$

3. $f(x) = \cos\sqrt{x}$

   $$f'(x) = -\sin(\sqrt{x})\frac{d}{dx}(\sqrt{x}) = -\sin(\sqrt{x})\cdot\frac{1}{2}(x)^{-\frac{1}{2}} = -\frac{\sin\sqrt{x}}{2\sqrt{x}}$$

4. $h(t) = (t^4-1)^3(t^3+1)^4$

   Need product rule, apply that 1st

   $h'(t) = (t^4-1)^3\frac{d}{dt}(t^3+1)^4 + (t^3+1)^4\frac{d}{dt}(t^4-1)^3$ and now chain

   $$= (t^4-1)^3 4(t^3+1)^3\frac{d}{dr}(t^3+1) + (t^3+1)^4 3(t^4-1)^2\frac{d}{dt}(t^4-1)$$

$$= (t^4-1)^3 4(t^3+1)^3(3t^2) + (t^3+1)^4 3(t^4-1)^2(4t^3)$$ and simplify

$$12t^2(t^3+1)^3(t^4-1)^2(t^4+t^2+t-1)$$

5. $G(y) = \left(\dfrac{y^2}{y+1}\right)^5$

Chain

$$G'(y) = 5\left(\frac{y^2}{y+1}\right)^4 \frac{d}{dy}\left(\frac{y^2}{y+1}\right)$$ need quotient rule now

$$= 5\left(\frac{y^2}{y+1}\right)^4\left[\frac{(y+1)\frac{d}{dy}(y^2) - (y^2)\frac{d}{dy}(y+1)}{(y+1)^2}\right]$$

$$= 5\left(\frac{y^2}{y+1}\right)^4\left[\frac{(y+1)(2y) - (y^2)(1)}{(y+1)^2}\right] = 5\left(\frac{y^2}{y+1}\right)^4\left[\frac{2y^2+2y-y^2}{(y+1)^2}\right] = 5\left(\frac{y^2}{y+1}\right)^4\left(\frac{y^2+2y}{(y+1)^2}\right)$$

$$5\left(\frac{y^2}{y+1}\right)^4\left(\frac{y(y+2)}{(y+1)^2}\right) = \frac{5y^8 y(y+2)}{(y+1)^6} = \mathbf{\frac{5y^9(y+2)}{(y+1)^6}}$$

Try on your own:

1. $y = \dfrac{\sin^2 x}{\cos x}$ $\qquad y' = 2+\sin x\cdot\tan^2 x$

2. $y = \sin^2(\cos 3x)$ $\qquad y' = -6\sin(\cos 3x)[\sin(3x)\cos(\cos 3x)]$

3. $y = (x^2+1)\sqrt[3]{x^2+2}$ $\qquad y' = 2x\left[\sqrt[3]{x^2+2} + \dfrac{1}{3}\dfrac{x^2+1}{(\sqrt[3]{x^2+2})^2}\right]$

4. $y = \sin(\sin(\sin x))$ $\qquad y' = \cos(\sin(\sin x))\cdot[\cos(\sin x)\cdot\cos x]$