Section 2.4 Product and Quotient Rules

1. Product Rule.
   If f and g are both differentiable:
   $$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}(g(x)) + \frac{d}{dx}(f(x))\cdot g(x)$$

   Alternatively, $(f\cdot g)' = f\cdot g' + g\cdot f'$

   What I say to myself: 1st times the derivative of the 2nd + 2nd times the derivative of the 1st.

2. Quotient Rule
   If f and g are both differentiable:
   $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot\frac{d}{dx}(f(x)) - f(x)\cdot\frac{d}{dx}(g(x))}{[g(x)]^2}$$

   Alternatively, $\left(\frac{f}{g}\right)' = \frac{g\cdot f' - f\cdot g'}{g^2}$

   Common Mnemonic: "(down d up – up d down) / down down"

   What I say to myself: denominator times derivative of the numerator – numerator times the derivative of the denominator by the denominator squared.

3. All of the trigonometric functions

   $$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc(x)\cdot\cot(x)$$
   $$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec(x)\cdot\tan(x)$$
   $$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

   Common Memory Tool: derivative of something starting with "c" is negative.

Examples:

1. $f(x) = \sqrt{x}\cdot\sin x$ (Product Rule)

$$f'(x) = \sqrt{x}\cdot\frac{d}{dx}(\sin x) + (\sin x)\cdot\frac{d}{dx}(\sqrt{x})$$
$$= \sqrt{x}\cdot\cos x + (\sin x)\cdot\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$
$$= \sqrt{x}\cdot\cos x + \frac{\sin x}{2\sqrt{x}}$$

2. $Y(u) = (u^{-2}+u^{-3})(u^5-2u^2)$

One way: $Y(u) = u^{-2}\cdot u^5 + u^{-2}(-2u^2) + u^{-3}\cdot u^5 + u^{-3}(-2u^2)$

$Y(u) = u^3 - 2 + u^2 - 2u^{-1}$

$Y'(u) = 3u^2 + 2u + 2u^{-2}$

Another way: $Y(u) = \underbrace{(u^{-2}+u^{-3})}_{1^{st}}\underbrace{(u^5-2u^2)}_{2^{nd}}$ (product rule)

$$Y'(u) = (u^{-2} + u^{-3})\cdot\frac{d}{du}(u^5 - 2u^2) + (u^5 - 2u^2)\cdot\frac{d}{du}(u^{-2}+u^{-3})$$

$$= (u^{02} + u^{-3})(5u^4 - 4u) + (u^5 - 2u^2)(-2u^{-3} - 3u^{-4})$$

$$= u^{-2}(5u^4) + u^{-2}(-4u) + u^{-3}(5u^4) + u^{-3}(-4u) + u^5(-2u^{-3}) + u^5(-3u^{-4}) + (-2u^2)(-2u^{-3}) + (-2u^2)(-3u^{-4})$$

$$= 5u^2 - \cancel{4u^{-1}} + 5u - 4u^{-2} - 2u^2 - 3u + \cancel{4u^{-1}} + 6u^{-2}$$

$$\mathbf{= 3u^2 + 2u + 2u^{-2}}$$

And, we have the same thing!

3. Show $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{(\cos x)\cdot\frac{d}{dx}(\sin x) - (\sin x)\cdot\frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$ Apply Pythagorean Identity

$$= \frac{1}{\cos^x} = \sec^2 x$$

4. Find $f'(x)$ when $f(x) = \dfrac{x}{x + \frac{c}{x}}$

Key is to simplify first: $f(x) = \dfrac{\frac{x}{1}}{\frac{\cancel{x^2+c}}{\cancel{x}}} \cdot \dfrac{\frac{x}{x^2+c}}{\frac{\cancel{x}}{\cancel{x^2+c}}} = \dfrac{x^2}{x^2+c}$

$$f'(x) = \frac{(x^2+c)\frac{d}{dx}(x^2) - (x^2)\frac{d}{dx}(x^2+c)}{(x^2+c)^2}$$

$$= \frac{(x^2+c)(2\mathrm{x}) - (x^2)(2\mathrm{x})}{(x^2+c)^2} = \frac{\cancel{2\mathrm{x}^3} + 2\mathrm{xc} - \cancel{2\mathrm{x}^3}}{(x^2+c)^2} = \mathbf{\frac{2cx}{(x^2+c)^2}}$$

5. Find equation of a tangent line to the curve $y = \dfrac{\sqrt{x}}{x+1}$ when $x = 4$.

$$y'(x) = \frac{(x+1)\frac{d}{dx}(\sqrt{x}) - (\sqrt{x})\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{(x+1)\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - (\sqrt{x})(1)}{(x+1)^2} = \frac{\frac{x+1}{2\sqrt{x}} - \sqrt{x}}{(x+1)^2}$$

Find the slope at $x = 4$:

$$y'(4) = \frac{\frac{4+1}{2\sqrt{4}} - \sqrt{4}}{(4+1)^2} = \frac{\frac{5}{2(2)} - 2}{25} = \frac{\frac{5}{4} - \frac{8}{4}}{25} = \frac{\frac{-8}{4}}{25} = -\frac{8}{100} = -0.08$$ Now we have a slope

I need a point:

$y(4) = \dfrac{\sqrt{4}}{4+1} = \dfrac{2}{5} = 0.4$ (4, 0.4)

Find the equation:

$$y - y_1 = m(x - x_1)$$
$$y - 0.4 = -0.08(x-4)$$