Section 3.6 Hyperbolic Functions

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$ hyperbolic sine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$ hyperbolic cosine

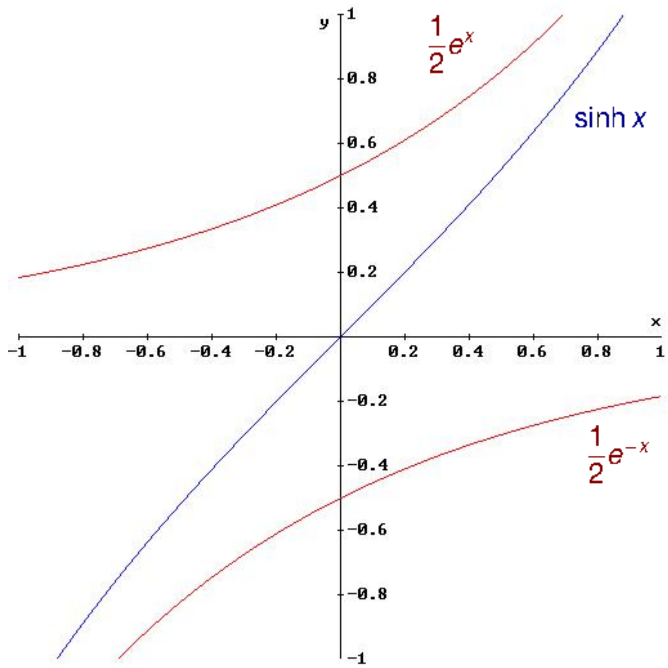
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$ hyperbolic tangent

$$\text{csch} = \frac{1}{\sinh x}$$ hyperbolic cosecant

$$\text{sech} = \frac{1}{\cosh x}$$ hyperbolic secant

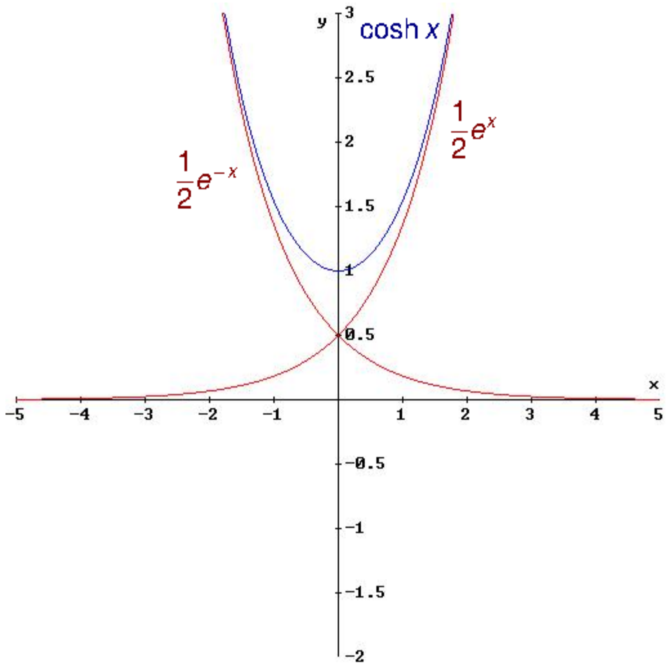
$$\coth = \frac{1}{\tanh x}$$ hyperbolic cotangent

The Graphs:



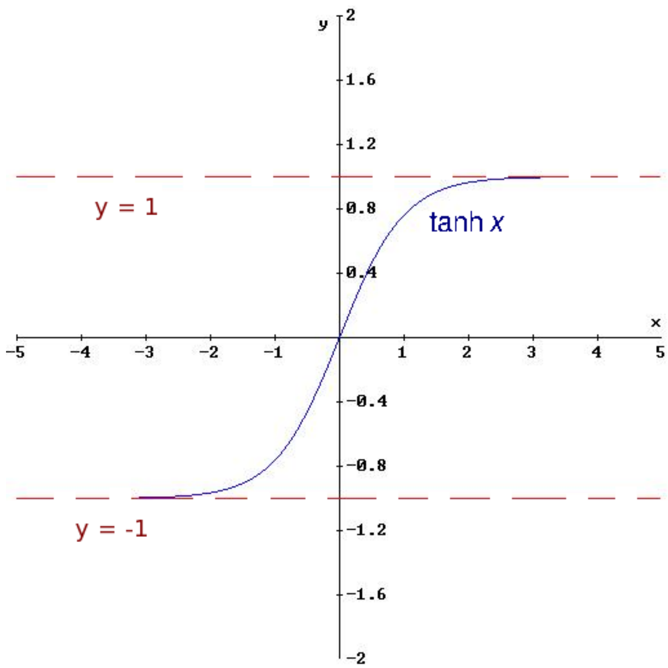
D: $x \in \mathbb{R}$
R: $y \in \mathbb{R}$

By Subtraction



D: $x \in \mathbb{R}$
R: $[1, \infty)$

By Addition



D: $x \in \mathbb{R}$
R: $(-1, 1)$

By Division

These functions are used heavily in science and engineering absorption, decay, and catenary problems.

Some handy identities (pg. 182)

1. $\sinh(-x) = -\sinh(x)$
2. $\cosh(-x) = \cosh(x)$
3. $\cosh^2(x) - \sinh^2(x) = 1$
4. $1 - \tanh^2(x) = \text{sech}^2(x)$

Hyperbolic Functions can also be differentiated.

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Example: Finding numerical values

pg. 185, #2 $\quad \tanh(0) = \dfrac{e^{(0)} - e^{(-0)}}{e^{(0)} + e^{(-0)}} = \dfrac{1-1}{1+1} = 0$

$$\tanh(1) = \frac{e^{(1)} - e^{(-1)}}{e^{(1)} + e^{(-1)}} = \frac{e - \frac{1}{e}}{e + \frac{1}{e}} = \frac{\frac{e^2-1}{e}}{\frac{e^2+1}{e}} = \frac{e^2-1}{e^2+1}$$

Example: Prove $\cosh(-x) = \cosh(x)$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^x}{2} = \cosh(x)$$

Example: Prove $\cosh x - \sinh x = e^{-x}$

$$\frac{e^x + e^{-x}}{2} - \left(\frac{e^x - e^{-x}}{2}\right) = \frac{\cancel{e^x} + e^{-x} - \cancel{e^x} + e^{-x}}{2} = \frac{\cancel{2} e^{-x}}{\cancel{2}} = e^{-x}$$

Example: If $\sinh x = \dfrac{3}{4}$, find the values of $\cosh x$ and $\tanh x$.

Use identities: $\cosh^2(x) - \sinh^2(x) = 1$

$$\cosh^2 x - \left(\frac{3}{4}\right)^2 = 1$$

So:

$$\cosh^2 x - \frac{9}{16} = 1$$

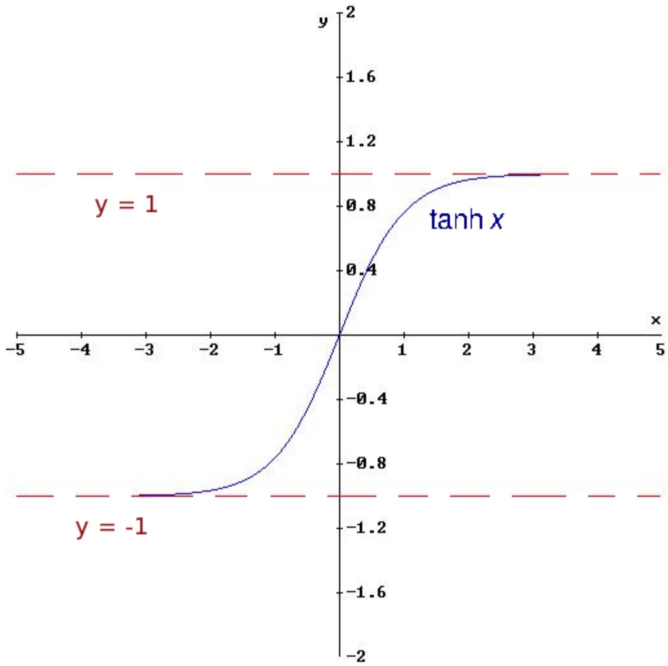
$$\cosh^2 x = \frac{25}{16}$$

$$\cosh x = \frac{5}{4}$$

And: $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{\frac{3}{4}}{\frac{5}{4}} = \dfrac{3}{5}$

Example: $\lim\limits_{x \to -\infty} \tanh x = ?$

By picture:



$\lim\limits_{x \to -\infty} \tanh x = -1$

Or, by algebra:

$$\lim_{x \to -\infty} \tanh x = \lim_{x \to -\infty} \frac{e^x - e^{-x}}{e^x + e^x} = \frac{\lim\limits_{x \to -\infty} e^x - \lim\limits_{x \to -\infty} e^{-x}}{\lim\limits_{x \to -\infty} e^x + \lim\limits_{x \to -\infty} e^{-x}}$$

$$= \frac{0 - \infty}{0 + \infty} = \frac{-\infty}{+\infty} \leftarrow \text{indeterminate form}$$

*We will learn how to handle this form in the next section (§3.7)

Example: Take the derivative of $g(x) = \sinh^2 x$

$$g(x) = (\sinh x)^2$$

$$g'(x) = 2(\sinh x)\frac{d}{dx}(\sinh x) \quad \text{chain rule still applies}$$

$$= 2(\sinh x)(\cosh x)$$

Example: Take derivative of $F(x) = \sinh(x)\tanh(x)$

$$F'(x) = \underbrace{\sinh x \frac{d}{dx}(\tanh x) + \tanh x \frac{d}{dx}(\sinh x)}_{\text{Product Rule}}$$

$$= (\sinh x)(\operatorname{sech}^2 x) + (\tanh x)(\cosh x)$$

Example: Take derivative of $f(t) = \ln(\sinh t)$

$$f'(t) = \frac{1}{\sinh t} \cdot \frac{1}{\ln e} \cdot \frac{d}{dt}(\sinh t)$$

$$= \frac{1}{\sinh t} \cdot \cosh t$$

$$f'(t) = \coth t$$

Example: Take derivative of $f(x) = \sinh\left[\cosh\left(\ln\left[\sec\left(x^2+1\right)\right]\right)\right]$

$$f'(x) = \cosh\left[\cosh\left(\ln\left[\sec\left(x^2+1\right)\right]\right)\right] \cdot \frac{d}{dx}\left[\cosh\left(\ln\left[\sec\left(x^2+1\right)\right]\right)\right]$$

$$= \cosh\left[\cosh\left(\ln\left[\sec\left(x^2+1\right)\right]\right)\right] \cdot \sinh\left(\ln\left[\sec\left(x^2+1\right)\right]\right) \cdot \frac{d}{dx}\left(\ln\left[\sec\left(x^2+1\right)\right]\right)$$

$$= \cosh\left[\cosh\left(\ln\left[\sec\left(x^2+1\right)\right]\right)\right] \cdot \sinh\left(\ln\left[\sec\left(x^2+1\right)\right]\right) \cdot \frac{1}{\sec\left(x^2+1\right)} \cdot \frac{d}{dx}\left(\sec\left(x^2+1\right)\right)$$

$$= \cosh\left[\cosh\left(\ln\left[\sec\left(x^2+1\right)\right]\right)\right] \cdot \sinh\left(\ln\left[\sec\left(x^2+1\right)\right]\right) \cdot \frac{1}{\sec\left(x^2+1\right)} \cdot$$
$$\sec\left(x^2+1\right) \cdot \tan\left(x^2+1\right) \cdot \frac{d}{dx}\left(x^2+1\right)$$

$$= \cosh\left[\cosh\left(\ln\left[\sec\left(x^2+1\right)\right]\right)\right] \cdot \sinh\left(\ln\left[\sec\left(x^2+1\right)\right]\right) \cdot \frac{1}{\cancel{\sec\left(x^2+1\right)}} \cdot$$
$$\cancel{\sec\left(x^2+1\right)} \cdot \tan\left(x^2+1\right) \cdot 2x$$

$$= 2x \tan\left(x^2+1\right) \sinh\left(\ln\left[\sec\left(x^2+1\right)\right]\right) \cosh\left[\cosh\left(\ln\left[\sec\left(x^2+1\right)\right]\right)\right]$$