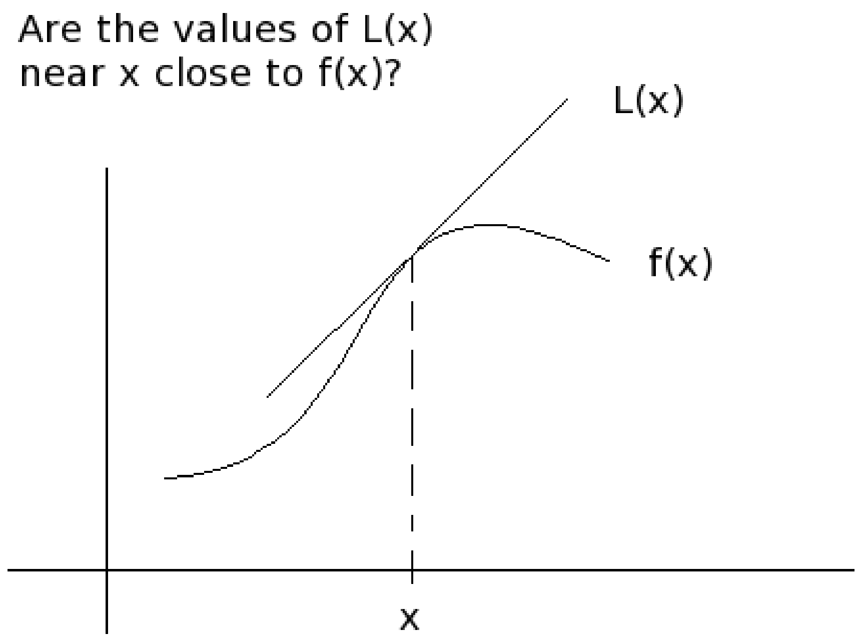
Section 2.8 Linear Approximations and Differentials

The idea is that we use a tangent line to approximate values close to some *x*.



Let $x = a$, then the point above is $(a, f(a))$

If I write out the equation of the tangent line through this point:

$$y - y_1 = f'(x)(x - x_1)$$ Basic Form

when $x = a$: $$y - f(a) = f'(a)(x - a)$$ Specific Form

Then just solve for y:

$$y = f(a) + f'(a)(x - a)$$

(or) $$f(x) \approx f(a) + f'(a)(x - a)$$

So, if we are close to $x = a$, this y might be a good approximation to f.

Traditionally, we rename the *y* as L(x)

$$L(x) = f(a) + f'(a)(x - a)$$

and call L(x) a linearization of f(x) near $x = a$.

Example: Find the linearization L(x) of the function at *a*:

$$f(x) = \frac{1}{\sqrt{2+x}} \qquad a = 0$$

1st Find *f ' (x):*

$$f(x) = (2+x)^{-\frac{1}{2}}$$

$$f'(x) = -\frac{1}{2}(2+x)^{-\frac{3}{2}}\frac{d}{dx}(2+x)$$

$$f'(x) = -\frac{1}{2}(2+x)^{-\frac{3}{2}}(1)$$

Simplify: $$f'(x) = \frac{-1}{2\sqrt{(2+x)^3}}$$

2nd $L(x) = f(a) + f'(a)(x-a)$ and fill in the pieces.

$$f(a) = f(0) = \frac{1}{\sqrt{2+0}} = \frac{1}{\sqrt{2}}$$

$$f'(a) = f'(0) = \frac{-1}{2\sqrt{(2+0)^3}} = \frac{-1}{2\sqrt{8}} = \frac{-1}{2\cdot 2\sqrt{2}} = \frac{-1}{4\sqrt{2}}$$

3rd Substitute:

$$L(x) = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}(x-0)$$

So, if I wanted an approximate linearized value for *x* = 0.01 (i.e. an estimate)

$$L(0.01) = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}(0.01)$$ would be a good guess

Example: Find the linear approximation of the function $g(x) = \sqrt[3]{1+x}$ at *a* = 0 and use it to approximate $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$

1st Find *g' (x):*

$$g(x) = (1+x)^{\frac{1}{3}}$$

$$g'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$$

$$g'(x) = \frac{1}{3\sqrt[3]{(1+x)^2}}$$

2nd $L(x) = f(a) + f'(a)(x-a)$

or $L(x) = g(a) + g'(a)(x-a)$

$$g(0) = \sqrt[3]{1+0} = \sqrt[3]{1} = 1$$

$$g'(0) = \frac{1}{3\sqrt[3]{(1+0)^2}} = \frac{1}{3(1)} = \frac{1}{3}$$

3rd Substitute:

$$L(x) = 1 + \frac{1}{3}(x-0)$$

$$L(x) = 1 + \frac{1}{3}x$$

4th Approximate

For our approximation : $\sqrt[3]{0.95}$ compare to $\sqrt[3]{1+x}$

$$0.95 = 1+x$$
$$x = -0.05$$

So: $L(x) = L(-0.05) = 1 + \frac{1}{3}(-0.05)$

$$= 1 + \frac{1}{3}\left(\frac{-5}{100}\right)$$
$$= 1 - \frac{5}{300} = \frac{295}{300}$$
$$= \frac{59}{60} \approx 0.9833$$

(Actual: $\sqrt[3]{0.95} = 0.9830$ Not Bad)

Likewise $\sqrt[3]{1.1}$ compare to $\sqrt[3]{1+x}$

$$1.1 = 1+x$$
$$x = 0.1$$

So: $L(x) = L(0.1) = 1 + \frac{1}{3}(0.1)$

$$= 1 + \frac{1}{3}\left(\frac{1}{10}\right)$$
$$= 1 + \frac{1}{30} = \frac{31}{30}$$
$$\approx 1.0\overline{3}$$

(Actual: $\sqrt[3]{1.1} = 1.0323$ Again, not bad)

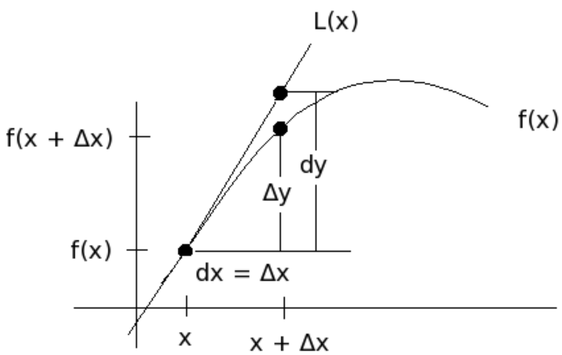
Differentials

$\frac{dy}{dx} = f'(x)$ notation

Key: $\underbrace{dy}_{\text{dependent}} = f'(x) \underbrace{dx}_{\text{independent}}$

* Small changes in *y* depend on the value of a derivative and a small change in *x*.

Think of it like this:



So: $dx = \Delta x$
$dy \neq \Delta y$

This *dy* is related through the tangent line not the function.

Notice that $\Delta y = f(x+\Delta x) - f(x)$
But, *dy* is related to the amount the tangent line deviates from *f* through a linearization
But notice that $dy \to \Delta y$ as $\Delta x \to 0$.
We can notationally change linearization to reflect this idea of differentials.

From Before:
$f(x) \approx f(a) + f'(a)(x-a)$

$dy = f'(x)\,dx$ Key: $x = a$: $dy = f'(a)\,dx$

Let: $dx = x - a$
$x = a + dx$

So: $f(a+dx) \approx f(a) + f'(a)(a+dx-a)$
$f(a+dx) \approx f(a) + f'(a)\,dx$

Concept: $f(a+dx) \approx f(a) + dy$

Example: The radius of a circular disk is given as 24 cm with a maximum error of 0.2 cm. (a) Use differentials to estimate the maximum error in the calculated area of the disk. (b) What is the relative error? (c) What is the percentage error?

(a)

$$A = \pi r^2$$
$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2)$$
$$\frac{dA}{dr} = 2\pi r$$
$$dA = 2\pi r\, dr$$
$$dA = 2\pi (24\,\text{cm})(0.2\,\text{cm})$$
$$dA = 9.6\pi\,\text{cm}^2 \quad (\approx\ 50.15\,\text{cm}^2)$$

(b)

$$\frac{dA}{A} = \frac{9.6\cancel{\pi}\,\text{cm}^2}{\cancel{\pi}\,24\,\text{cm}^2} = 0.4$$ (about twice that of the radius)

(c) Just multiply by 100

0.4 * 100 = 40%

More Examples:

1. Approximate $\sqrt{99.8}$

Notice that $\sqrt{100} \approx \sqrt{99.8}$

$\sqrt{100-x}$ where x = 0.02 makes $\sqrt{99.8}$

Choose $f(x) = \sqrt{100-x} = (100-x)^{\frac{1}{2}}$

Linearize around a = 0

$$f'(x) = \frac{1}{2}(100-x)^{-\frac{1}{2}}(-1)$$

$$f(a) = f(0) = \sqrt{100-0} = 10$$

$$f'(a) = f'(0) = -\frac{1}{2}(100-0)^{-\frac{1}{2}} = -\frac{1}{2\cdot 10} = -\frac{1}{20}$$

$$L(x) = f(a) + f'(a)(x-a)$$
$$L(x) = 10 - \frac{1}{20}x$$

$x = 0.2$

$$L(0.2) = 10 - \frac{1}{20}\left(\frac{2}{10}\right) = 10 - \frac{1}{100} = 9.99$$

2. Approximate $(8.06)^{\frac{2}{3}} = \sqrt[3]{(8.06)^2}$

Notice that: $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = \sqrt[3]{4^3} = 4$

and $8^{\frac{2}{3}} \approx (8.06)^{\frac{2}{3}}$

Choose: $f(x) = (8+x)^{\frac{2}{3}}$ $x = 0.06$

Linearize around $a = 0$.

$$f'(x) = \frac{2}{3}(8+x)^{-\frac{1}{3}}(1)$$

$$f(0) = (8+0)^{\frac{2}{3}} = 8^{\frac{2}{3}} = 4$$

$$f'(0) = \frac{2}{3}(8+0)^{-\frac{1}{3}} = \frac{2}{3(2)} = \frac{1}{3}$$

Then:

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = f(0) + f'(0)(x-0)$$

$$L(x) = 4 + \frac{1}{3}x$$

$$L(0.06) = 4 + \frac{1}{3}\left(\frac{6}{100}\right)$$

$$= 4 + \frac{2}{100}$$

$$= 4.02$$

So: $(8.06)^{\frac{2}{3}} \approx 4.02$

3. (From Homework) 21. The edge of cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use the differentials to estimate the maximum possible error, relative error, and percentage error when computing: (a) the volume of the cube and (b) the surface area of the cube.

(a) The Volume of the Cube:

$x = 30 \text{ cm} \pm 0.1 \text{ cm}$

$$V = x^3$$
$$\frac{dV}{dx} = 3\text{x}^2$$
$$dV = 3\text{x}^2 dx$$
$$dV = 3(30\,\text{cm})^2(\pm 0.1\,\text{cm})$$
$$dV = 3(900\,\text{cm}^2)(\pm 0.1\,\text{cm})$$
$$dV = 2700\,\text{cm}^2(\pm 0.1\,\text{cm})$$
$$dV = \pm 270\,\text{cm}^3$$

Relative Error: Divide the error by the total volume:

$$\frac{dV}{V} = \frac{\pm 270\,\text{cm}^3}{(30\text{ cm})^3} = \frac{\pm 270\,\text{cm}^3}{27{,}000\,\text{cm}^3} = 0.01$$

Percentage Error: multiply by 100: 0.01 * 100 = 1%

(b) The Surface Area of the Cube:

$$A = 6\text{x}^2$$
$$\frac{dA}{dx} = 6(2\text{x})$$
$$dA = 12\text{x}\, dx$$
$$dA = 12(30\,\text{cm})(\pm 0.1\,\text{cm})$$
$$dA = \pm 36\,\text{cm}^2$$

Relative Error: Divide the error by the total area:

$$\frac{dA}{A} = \frac{\pm 36\,\text{cm}^2}{6\,(30\text{ cm})^2} = \frac{\pm 36\,\text{cm}^2}{5{,}400\,\text{cm}^2} = 0.067$$

Percentage Error: multiply by 100: 0.0067 * 100 = 0.67%

4. The circumference of a sphere was measured to be 84 cm with possible error of 0.5 cm. (a) Use differentials to estimate maximum error calculated in surface area. (b) Use differentials to estimate maximum error in calculating volume.

(a) Use differentials to estimate maximum error calculated in surface area.

$$C = 2\pi r \qquad\qquad \frac{dC}{dr} = 2\pi$$
$$84\,\text{cm} = 2\pi r \qquad \text{And} \qquad dC = 2\pi\, dr$$
$$r = \frac{42}{\pi}\,\text{cm} \qquad\qquad 0.5\,\text{cm} = 2\pi\, dr$$

$$A = 4\pi r^2 \qquad\qquad A = 4\pi\left(\frac{42}{\pi}\,\text{cm}\right)^2$$
$$\frac{dA}{dr} = 4\pi(2\text{r}) \qquad\qquad A = \frac{7056}{\pi}\,\text{cm}^2$$
$$dA = 4\text{r}(2\pi\, dr)$$

Maximum Error:

$$dA = 4\left(\frac{42}{\pi}\,\text{cm}\right)(0.5\,\text{cm})$$
$$dA = \frac{84}{\pi}\,\text{cm}^2$$

Relative Error:

$$\frac{dA}{A} = \frac{\frac{84}{\cancel{\pi}}}{\frac{7056}{\cancel{\pi}}} = \frac{84}{7056} = 0.0119$$

(b) Use differentials to estimate maximum error in calculating volume.

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dr} = \frac{4}{\cancel{3}}\cdot\cancel{3}\pi r^2$$
$$dV = 2r^2(2\pi\, dr)$$

Maximum Error:

$$dV = 2\left(\frac{42}{\pi}\,\text{cm}\right)^2(0.5\,\text{cm})$$
$$dV = \frac{1764}{\pi^2}\,\text{cm}^3$$

Relative Error:

$$\frac{dV}{V} = \frac{\dfrac{1764}{\pi^2}\,\mathrm{cm}^3}{\dfrac{4}{3}\pi\left(\dfrac{42}{\pi}\,\mathrm{cm}\right)^3} = \frac{\dfrac{1764}{\cancel{\pi^2}}}{\dfrac{78{,}784}{\cancel{\pi^2}}} = \frac{1}{56} = 0.0179$$