Section 2.3 Basic Differentiation Rules

We do not always want to use the limit form of the definition of the derivative to find derivatives (too painful). So, over the centuries, “shortcut” rules have been found for calculating them. This section is all about learning some of those rules.

1. Constants: $\frac{d}{dx}(c) = 0$ where c is any constant.

   Example: $\frac{d}{dx}(3) = 0$

2. Power Functions: $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$ where n is any real number.

   Example: $\frac{d}{dx}(x^6) = 6 \cdot x^5$ $\qquad$ $\frac{d}{dx}(x) = x^0 = 1$

   Example: $\frac{d}{dx}\left(\frac{1}{x^6}\right) = \frac{d}{dx}(x^{-6})$ rewrite $\frac{d}{dx}(x^{-6}) = -6 \cdot x^{-7}$

   Example: $\frac{d}{dx}\left(\sqrt[3]{x^7}\right) = \frac{d}{dx}\left(x^{\frac{3}{7}}\right)$ rewrite $\frac{d}{dx}\left(x^{\frac{3}{7}}\right) = \frac{7}{3} \cdot x^{\frac{4}{3}}$

   Example: $\frac{d}{dx}\left(\frac{x}{\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{x}{x^{1/2}}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

3. <u>Constant Multiple:</u> $\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}(f(x))$ where c is a constant.

   Example: $\frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2) = 3(2x) = 6x$

4. <u>The Sum Rule:</u> $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$.

5. <u>The Difference Rule:</u> $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$.

   Example:
   $$\frac{d}{dx}(x^5 + 4x^3 - x^2 - 1) = \frac{d}{dx}(x^5) + 4\frac{d}{dx}(x^3) - \frac{d}{dx}(x^2) - \frac{d}{dx}(1)$$
   $$= 5x^4 + 4(3x^2) - 2x - 0 = \mathbf{5x^4 + 12x^2 - 2x}$$

6. <u>The Trigonometric Rules:</u> $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$

Example: $\frac{d}{d\theta}(5\cos\theta) = 5\frac{d}{d\theta}(\cos\theta) = -5\sin\theta$

Examples

1. Find $R'(x)$ when $R(x) = \frac{\sqrt{10}}{x^7}$.

$$R(x) = \frac{\sqrt{10}}{x^7} = \sqrt{10}(x^{-7})$$

$$R'(x) = \frac{d}{dx}\left(\sqrt{10}(x^{-7})\right) = \sqrt{10}(-7)x^{-8} = \frac{-7\sqrt{10}}{x^8}$$

2. Find $g'(u)$ when $g(u)=\sqrt{2}\,u+\sqrt{3u}$.

$$g'(u) = \frac{d}{du}\left(\sqrt{2}\,u\right) + \frac{d}{du}\left(\sqrt{3u}\right) = \sqrt{2}\frac{d}{du}(u) + \frac{d}{du}(3u)^{\frac{1}{2}} = \sqrt{2}(1) + \sqrt{3}\frac{1}{2}u^{-\frac{1}{2}}$$

$$= \sqrt{2}+\frac{d}{du}\left(3^{\frac{1}{2}}\cdot u^{\frac{1}{2}}\right) = \frac{d}{du}\left(\sqrt{3}\cdot u^{\frac{1}{2}}\right)$$

$$g'(u) = \sqrt{2} + \frac{\sqrt{3}}{2\sqrt{u}}$$

3. Find y' when $y = \frac{x^2-2\sqrt{x}}{x}$.

$$y = \frac{x^2}{x} - \frac{2\sqrt{x}}{x} = x - \frac{2}{\sqrt{x}} = x - 2x^{-\frac{1}{2}}$$

$$y' = 1-2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} = 1 + \frac{1}{\sqrt{x^3}}$$

4. For what values of x does the graph of $f(x) = x^3 + 2x^2 + x - 3$ have a horizontal tangent?
Note: horizontal tangent means $f'(x) = 0$.

Find $f'(x)$: $f'(x) = 3x^2 + 4x + 1$

$$f'(x) = (3x + 1)(x + 1) = 0 \Rightarrow \quad 3x + 1 = 0 \quad \text{and} \quad x + 1 = 0$$

$$x = -\frac{1}{3} \quad \text{and} \quad x = -1$$

Now, part 2. What is the equation of the tangent line when x = 2?
The slope at x = 2 is $f\,'(2)$

$$f\,'(x) = 3x^2 + 4x + 1$$

$$f\,'(2) = 3(2)^2 + 4(x) + 1 = 3(4) + 4(2) + 1 = 12 + 8 + 1 = \mathbf{21}$$

The point on $f(x)$ when x = 2 is $(2, f(2))$

$$f(x) = x^3 + 2x^2 + x - 3$$

$$f(2) = 2^3 + 2(2)^2 + 2 - 3 = 8 + 8 + 2 - 3 = \mathbf{15}$$

So, (2, 15) is on $f(x)$.

We have a point (2, 15) and a slope m = 21. Apply point-slope equation of a line.

$$y - y_1 = m(x - x_1)$$
$$y - 15 = 21(x - 2)$$

$y = 12x - 42 + 15 = \mathbf{21x - 27}$ <- equation of a tangent line at x = 2.

5. Equation of position given as:

$$s(t) = 2t^3 - 7t^2 + 4t + 1$$

a. Find velocity and acceleration as a function of time:

$$v(t) = s'(t) = 2(3t^2) - 7(2t) + 4(1) + 0 = 6t^2 - 14t + 4 \quad \left[\frac{m}{s}\right]$$

$$a(t) = v'(t) = 6(2t) - 14(1) = 8t - 14 \quad \left[\frac{m}{s^2}\right]$$

b. What is the acceleration after 1 second?
i.e. what is $a(1)$?

$$a(1) = 8(1) - 14 = -6 \ \frac{m}{s^2}$$

c. What is the acceleration when the velocity is zero?

$v(t) = 0$
$$6t^2 - 14t + 4 = 0$$
$$3t^2 - 7t + 2 = 0$$
$$(3t - 1)(t - 2) = 0$$

$3t - 1 = 0 \qquad \text{and} \qquad t - 2 = 0$

$t = \frac{1}{3} \qquad \text{and} \qquad t = 2$

$$a\left(\frac{1}{3}\right) = 8\left(\frac{1}{3}\right) - 14 = \frac{8}{3} - \frac{42}{3} = -\frac{34}{3} \ \text{ or } \ -11.\overline{3} \quad \frac{m}{s^2}$$

$$a(2) = 8(2) - 14 = 16 - 14 = 2 \quad \frac{m}{s^2}$$