Section 1.5 Continuity

We've actually already defined what it means for a function to be continuous when we started talking about limits.

Definition: A function, *f*, is continuous at a number if:

$$\lim_{x \to a} f(x) = f(a)$$

Notice, that this definition means:

1. *f(a)* is defined (*a* is in the domain & *f(a)* = value)
2. $\lim_{x \to a} f(x)$ exists
3. that the limit at *a* equals function value at *a*

*BUT, DON'T EVER FORGET: limits and function values are not the same thing!

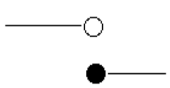
Alternatively, what does it mean for a function to be discontinuous at *a*?

$$\lim_{x \to a} f(x) \neq f(a)$$

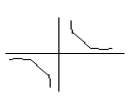
What does that mean? What could cause that?

Holes: $\lim_{x \to a} f(x) = value$ but *f(a)* is undefined

Jumps $\lim_{x \to a} = \text{DNE}$ but *f(a)* = value



Asymptotes $\lim_{x \to a} = \text{DNE}$ and *f(a)* is undefined



How do we tell what kind of discontinuity we might have in a function? Using limit relationships.

Example: $f(x) = \dfrac{x^2 - x - 12}{x+3}$

We can look at this denominator x + 3 = 0 and set it to zero.

That means we have a disc at x = -3. But, what kind?
Evaluate the limit there:

$$\lim_{x \to 3} \frac{x^2 - x - 12}{x + 3} = \lim_{x \to 3} \frac{(x-4)\cancel{(x+3)}}{\cancel{(x+3)}} = -3 - 4 = -7$$

So, $\lim_{x \to a} f(a) = \text{value}$, but *f(a)* is undefined.

Hence, a HOLE (terminology, Removable Discontinuity)

Example: $f(x) = \dfrac{x^2 - x}{x^2 - 1}$

Look at denominator:

$$(x^2 - 1) = 0$$
$$(x+1)(x-1) = 0$$
$$x = -1 \qquad x = 1$$

Discontinuity at $x = -1$ and $x = 1$, but what kind?

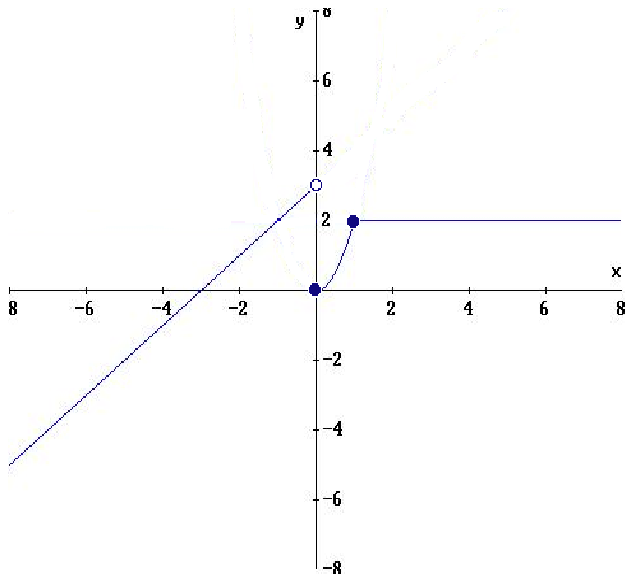
$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x\,\cancel{(x-1)}}{\cancel{(x-1)}\,(x+1)} = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2}$$

So, the limit has a value, but the function does not. This is a HOLE.

$$\lim_{x \to -1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to -1} \frac{x\,\cancel{(x-1)}}{\cancel{(x-1)}\,(x+1)} = \lim_{x \to -1} \frac{x}{x+1} = \text{DNE}$$

Nothing can be done. So, the limit does not exists and the function does not exist.
This means $x = -1$ is an asymptote.

Example: $f(x) = \begin{cases} x+3 & x < 0 \\ 2x^2 & 0 \le x < 1 \\ 2 & x > 1 \end{cases}$



By picture:

$\lim_{x \to 0} f(x) = \text{DNE}$ but f(0) = 0 Jump

$\lim_{x \to 0^-} f(x) = 3$ $\lim_{x \to 0^+} f(x) = 0$ continuous from left and right

Definition: Continuous from the right

$$\lim_{x \to a^+} f(x) = f(a)$$

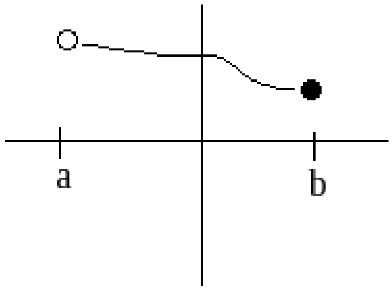
Continuous from the left

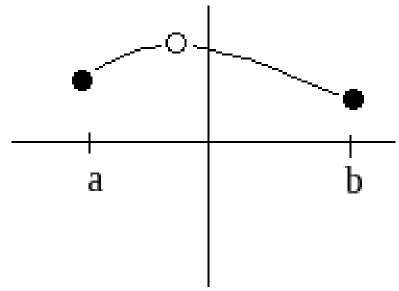
$$\lim_{x \to a^-} f(x) = f(a)$$

So, our last example was continuous from the right.

Definition: Continuous on an interval

Continuous from left to right but maybe not end points.





Continuous on (a , b)
Continuous on (a , b]
Not continuous on [a , b) or [a , b]

Not continuous on any form of (a , b) , [a , b]

Theorem: Manipulating Continuous Functions:
If *f* is continuous and *g* is continuous at *a* and *c = constant*, the following are also continuous at *a*.

1. $f + g$
2. $f - g$
3. $c*f$ *or* $c*g$
4. $f*g$
5. $\frac{f}{g}$ $g \neq 0$

Theorem:

- Polynomials are continuous everywhere
- Rational functions are continuous everywhere except when denominator = 0
- Root functions are continuous everywhere in their domains
- Trigonometric functions are continuous everywhere in their domains.

Example: Where is $h(x) = \sqrt[3]{x} \cdot (1+x^3)$ continuous?

To proceed, evaluate the domain of each "piece" of the function.

Root: Let $f(x) = \sqrt[3]{x}$ *D: f(x):* $x \in \mathbb{R}$

Poly: Let $g(x) = 1+x^3$ *D: g(x):* $x \in \mathbb{R}$

Since, *h(x) = g(x) f(x),*
And *f(x)* and *g(x)* are continuous in their full domains, *h(x)* is also continuous on

$x \in \mathbb{R}$

Example: Where is $h(x) = \dfrac{\sin x}{x+1} + \sqrt{x-1}$ continuous?

Trig: Let $f(x) = \sin x$ *D: f(x):* $x \in \mathbb{R}$

Poly: Let $g(x) = x+1$ *D: g(x):* $x \in \mathbb{R}$ but $\dfrac{f(x)}{g(x)} \neq 0 \Rightarrow x \neq -1$

Root: Let $p(x) = \sqrt{x-1}$ *D: p(x):* $x \in [1, \infty)$

Since: $h(x) = \dfrac{f(x)}{g(x)} + p(x)$ *h(x)* continuous on $x \in [1, \infty)$

Theorem: If *f* is continuous at *b*, and $\lim_{x \to a} g(x) = b$ then

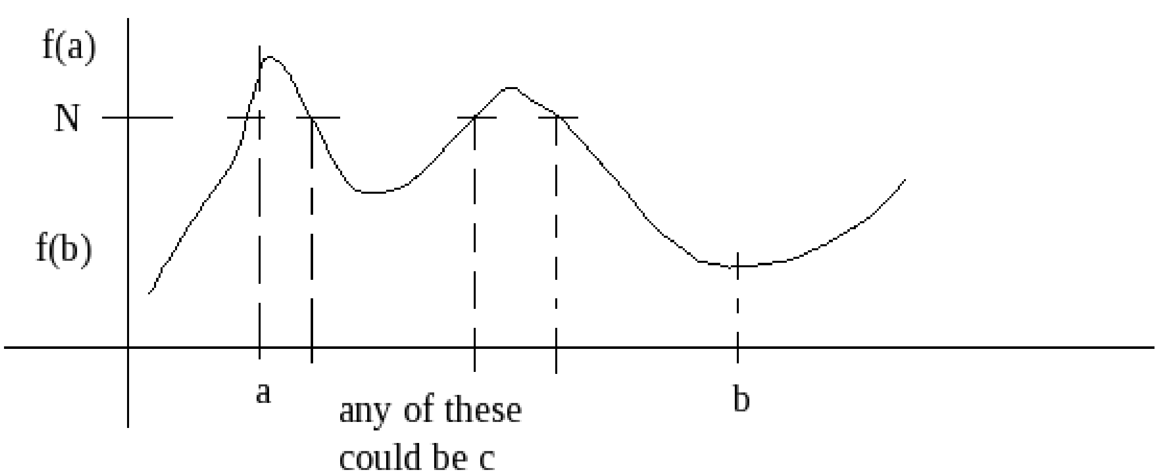
$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b)$$

Basically, the composition of two continuous functions is continuous.

BIG ONE!

(pg. 52) **Intermediate Value Theorem** (IVT)

Suppose that *f* is continuous on the closed interval [*a* , *b*] and let *N* be any number between *f(a)* and *f(b)*, where *f(a)* ≠ *f(b)*. Then, there exists a number *c* in [*a* , *b*] such that *f(c)* = *N*.



The IVT is used primarily in engineering to prove that roots of functions exist.

Example: Show that cos $x = x$ has a root on the interval (0 , 1).

1st, Let $f(x) = \cos x - x$

Now, show that *f(x) = 0* somewhere on the interval $x \in (0,1)$

2nd, find
$$f(0) = \cos(0) - 0 = 1 - 0 = 1$$
$$f(1) = \cos(1) - 1 \approx 0.5403 - 1 = -0.4597$$

3rd, thus: *f(0)* > 0 > *f(1)*
1 > 0 > -0.4597

That means $N = 0$ is a number between *f(0)* and *f(1)*. *f* is continuous because it's a combination of trigonometric function and a polynomial. So IVT holds. So, there must be a "c" between 0 and 1 such that *f(c)* = 0.

In other words, *f(x)* = cos $x - x$ must have at least 1 root between $x \in (0,1)$.

Aside, how might you estimate the root?
Hint, apply the IVT iteratively while making the interval smaller each time.