Section 3.2 Inverse Functions and Logarithms

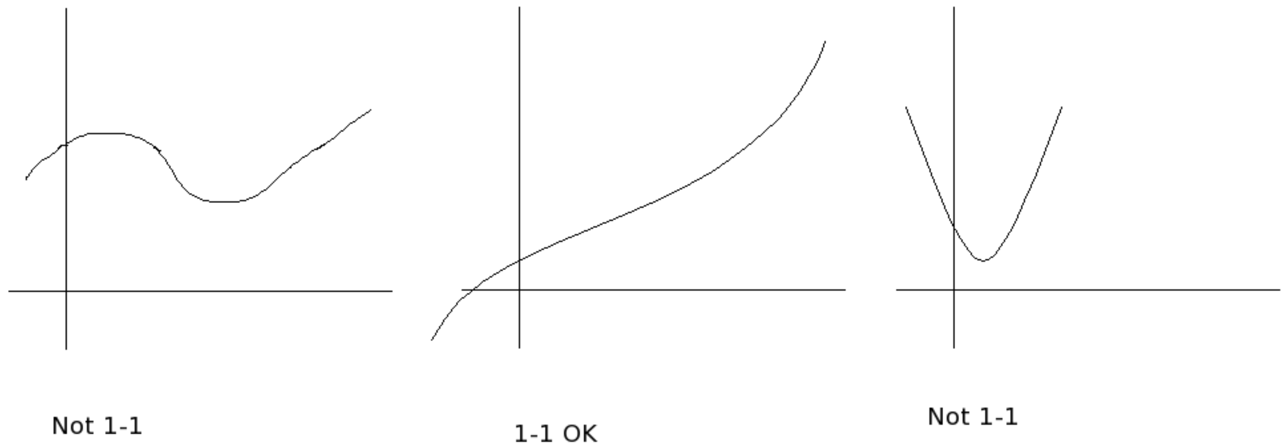
Definition: A function, *f*, is called a one-to-one function if it never takes on the same value twice.

$$f(x_1) \neq f(x_2) \quad \text{when} \quad x_1 \neq x_2$$

Horizontal Line Test (Not to be confused with the Vertical Line Test)

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Examples:

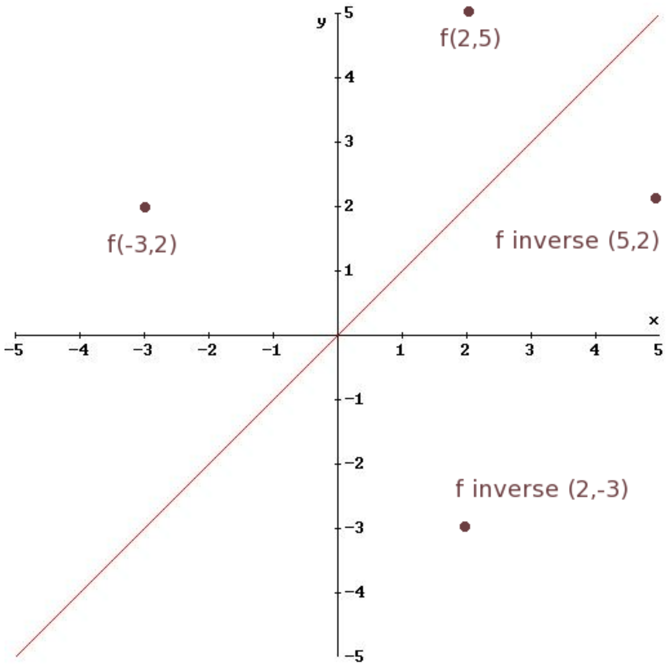


Definition: Let *f* be a one-to-one function with domain *A* and range *B*. Then its inverse function, f^{-1} has domain *B* and range *A* as is defined by: $f^{-1}(y) = x \Rightarrow f(x) = y$.

Example: Say you know that *f(x)* is one-to-one and $f(2)=5$, $f(-3)=2$, and $f(11)=7$ then you also know: $f^{-1}(5)=2$, $f^{-1}(2)=-3$, and $f^{-1}(7)=11$

*Essentially, the *x*'s and *y*'s change places

Notice that the points are reflections of each other over the line $y = x$.



How do you figure out the inverse function of a one-to-one function?

1. Write out $y = f(x)$
2. Solve the entire expression for x in terms of y
3. Recast as $f^{-1}(x)$

Example: Find $f^{-1}(x)$ of $f(x) = \dfrac{4x-1}{2x+3}$

1. $$y = \frac{4x-1}{2x+3}$$
$$y(2x+3) = 4x-1$$
$$2xy+3y = 4x-1$$

2. $$2xy-4x = -3y-1$$
$$x(2y-4) = -3y-1$$
$$x = \frac{-3y-1}{2y-4}$$

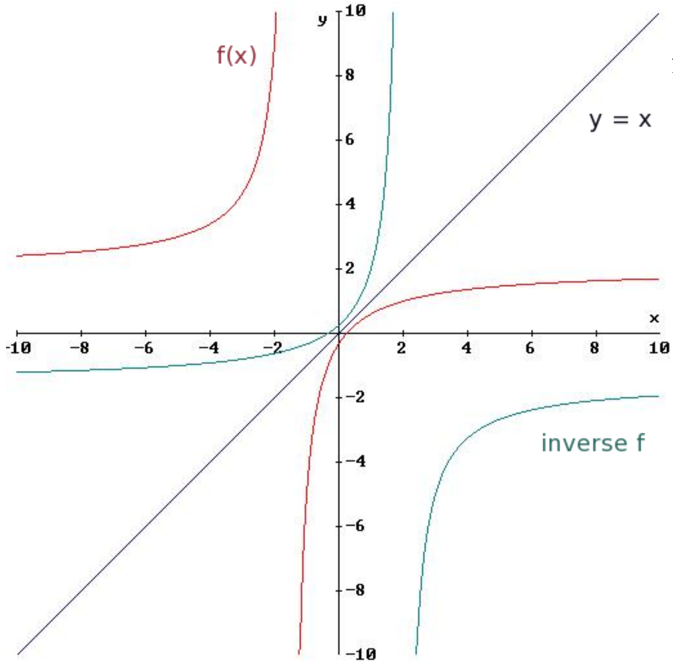
3. $$f^{-1}(x) = \frac{-3x-1}{2x-4}$$

Test Point: $f(1) = \dfrac{4(1)-1}{2(1)+3} = \dfrac{3}{5}$ $\left(1\ ,\ \dfrac{3}{5}\right)$

$$f^{-1}\left(\frac{3}{5}\right) = \frac{-3\left(\frac{3}{5}\right)-1}{2\left(\frac{3}{5}\right)-4} = \frac{\frac{-9}{5}-\frac{5}{5}}{\frac{6}{5}-\frac{20}{5}} = \frac{\frac{-14}{5}}{\frac{-14}{5}} = 1$$ $\left(\dfrac{3}{5}\ ,\ 1\right)$ OK

What do the graphs look like?

$$f(x)=\frac{4x-1}{2x+3} \quad \text{And} \quad f^{-1}(x)=\frac{-3x-1}{2x-4}$$



f(x):

R: $y\in(-\infty,2)\cup(2,\infty)$

D: $x\in\left(-\infty,-\frac{3}{2}\right)\cup\left(-\frac{3}{2},\infty\right)$

H.A. @ $y=\frac{4}{2}=2$

V.A. @ $x=-\frac{3}{2}$

Test Points:		
	(-1.6)	- / - = +
	(-1.5)	undefined
	(-1.4)	- / + = -

$f^{-1}(x)$:

R: $y\in\left(-\infty,-\frac{3}{2}\right)\cup\left(-\frac{3}{2},\infty\right)$

D: $x\in(-\infty,2)\cup(2,\infty)$

H.A. @ $y=-\frac{3}{2}$

V.A. @ $x=\frac{4}{2}=2$

So, sketches can be very easy. We just flip over $y = x$.

Handy Theorems:

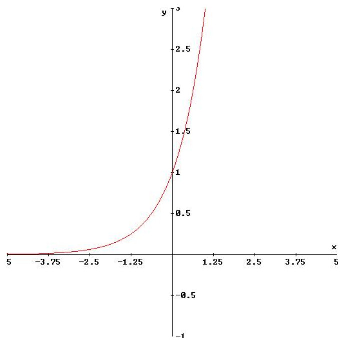
1. If f is one-to-one and continuous, then its inverse function f^{-1} is also continuous.
2. If f is one-to-one and differentiable with f^{-1} and $f'\left(f^{-1}(a)\right)$ then:

$$\left(f^{-1}\right)'(a)=\frac{1}{f'\left(f^{-1}(a)\right)}$$

<u>Logarithmic</u> <u>Functions</u>

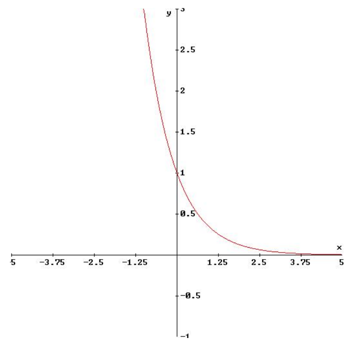
$f(x) = a^x$

If $a > 0$ and $a \neq 1$



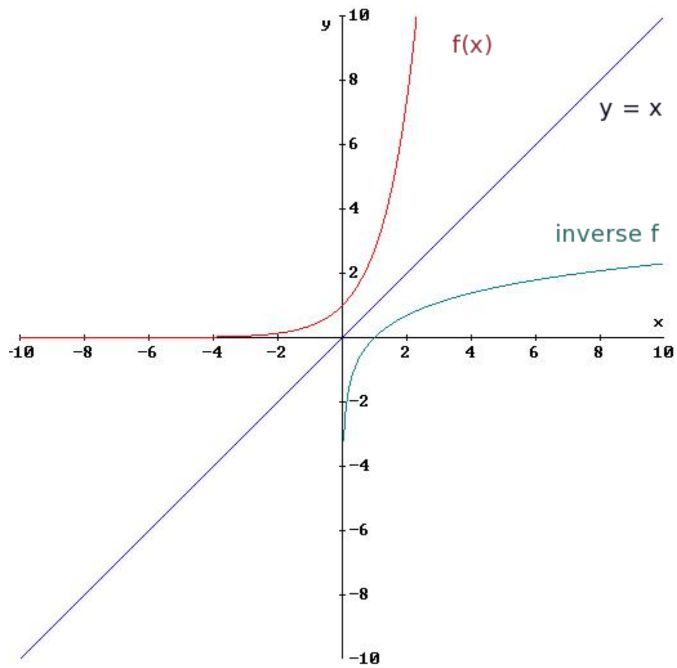
Increasing

If $a < 1$



Decreasing

By horizontal line test, both are one-to-one. Hence, each one has an inverse function.



$f(x) = a^x \qquad a > 1$

Notation: $\log_a x = y \quad \leftarrow\rightarrow \quad a^y = x$

Example: $\log_2 8 = 3 \qquad 2^3 = 8$

What is the exponent I need to raise 2 to, to get the answer 8?

Logarithmic Properties

1. $\log_a \left(a^x\right) = x$ For every $x \in \mathbb{R}$

   Read as, what is the exponent I need to raise *a* to, to get a^x ? x

2. $a^{\log_a x} = x$

All logarithms that look like $\log_a(x)$ have domain $(0, \infty)$ and range $(-\infty, \infty)$

Logarithmic Laws

1. $\log_a(x \cdot y) = \log_a x + \log_a y$

2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a\left(x^r\right) = r \log_a x$ $r \in \mathbb{R}$

Example: Expand $\log_3 \sqrt{a\left(b^2+c^2\right)}$

$$\log_3 \sqrt{a\left(b^2+c^2\right)} = \log_3\left[a\left(b^2+c^2\right)\right]^{\frac{1}{2}} = \frac{1}{2}\log_3\left[a\left(b^2+c^2\right)\right] = \frac{1}{2}\left[\log_3(a) + \log_3\left(b^2+c^2\right)\right]$$

Example: Express the quantity $\log_2 x + a\log_2 y - b\log_2 z$ as a single logarithm.

$$\log_2 x + a\log_2 y - b\log_2 z = \log_2 x + \log_2 y^a - \log_2 z^b = \log_2\left(\frac{xy^a}{z^b}\right)$$

Example: Evaluate: $\log_5 10 + \log_5 20 - 3\log_5 2$

$$\log_5 10 + \log_5 20 - \log_5 2^3 = \log_5\left(\frac{10 \cdot 20}{2^3}\right)$$

$$= \log_5\left(\frac{\cancel{2} \cdot 5 \cdot \cancel{2} \cdot \cancel{2} \cdot 5}{\cancel{2} \cdot \cancel{2} \cdot \cancel{2}}\right) = \log_5(25) = \mathbf{2}$$

What do I have to raise 5 to, to get 25?

Some other notes:

$$\log_e x = \ln x \quad \Leftrightarrow \quad \ln x = y \; , \; e^y = x$$

Inverse Properties

$$\ln\left(e^{x}\right) = \log_{e}\left(e^{x}\right) = x \qquad x \in \mathbb{R}$$

$$e^{\ln(x)} = x \qquad x > 0$$

Change of Base Formula

$$\log_{a} x = \frac{\log_{b} x}{\log_{b} a}$$ (Remember, "The base goes to the basement")

This means we can turn any logarithm into another logarithm.

Why: $y = \log_{a} x$

So, we know: $a^{y} = x$

Taking the logarithms of both sides: $\log_{b} a^{y} = \log_{b} x$

Using the power rule: $y \log_{b} a = \log_{b} x$

Solve for *y*: $y = \dfrac{\log_{b} x}{\log_{b} a}$

*Best to choose $b = 10$ or $b = e$. Why? Because those logarithms are on our calculator.

Example: Evaluate: $\log_{3} 15$

$$\log_{3} 15 = \frac{\log_{10} 15}{\log_{10} 3} \approx 2.456$$

$$= \frac{\ln 15}{\ln 3} \approx 2.465$$

As previously mentioned:

Theorem: If f is one-to-one and differentiable with inverse function f^{-1} and $f'\left(f^{-1}(a)\right) \neq 0$ then the inverse function is differentiable at *a* and:

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$$

Why? $\left(f^{-1}\right)'(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a}$

If $f(b) = a$ then we know $f^{-1}(a) = b$

so, then $f(y) = x$ and let $f^{-1}(x) = y$

As $x \to a$, $f^{-1}(x) \to f^{-1}(a)$
$y \to b$

Put this back into our formula

$$\left(f^{-1}\right)'(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a} = \lim_{y \to b} \frac{y - b}{f(y) - f(b)} = \lim_{y \to b} \frac{1}{\frac{f(y) - f(b)}{y - b}} = \frac{1}{\lim_{y \to b} \frac{f(y) - f(b)}{y - b}}$$

By definition: $= \frac{1}{f'(b)}$ and replace b

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$$

Example: Solve $e^{2x+3} - 7 = 0$

$$e^{2x+3} = 7$$
$$\log_e\left(e^{2x+3}\right) = 7$$
$$2x+3 = \ln(7)$$

$$2x = \ln(7)-3$$
$$x = \frac{\ln(7)-3}{2}$$

Example: Solve $\ln(5-2x) = -3$

$$e^{\ln(5-2x)} = e^{-3}$$
$$5-2x = e^{-3}$$
$$-2x = e^{-3}-5$$
$$x = \frac{e^{-3}-5}{-2} = \frac{5-e^{-3}}{2}$$