Section 1.4 Calculating Limits

First, the rules:

Suppose: $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

Then:

Summation: $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

Difference: $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

Constant over: $\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x)$
a function

Product: $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

Quotient: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided $\lim_{x \to a} g(x) \neq 0$

Power: $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$

Constant: $\lim_{x \to a} c = c$

Some Basic Tools:

Basis for all polynomials: $\lim_{x \to a} x = a$

n, a positive integer: $\lim_{x \to a} x^n = a^n$

n, a positive integer: $\lim_{x \to a} \sqrt[n]{x} = \lim_{x \to a} x^{\frac{1}{n}} = \sqrt[n]{a} = a^{\frac{1}{n}}$
with $a > 0$, n = even a unrestricted, n = odd

Putting it all together: $\lim_{x \to a} \sqrt[n]{f(x)} = \lim_{x \to a} [f(x)]^{\frac{1}{n}} = \left[\lim_{x \to a} f(x)\right]^{\frac{1}{n}} \overset{\text{or}}{=} \sqrt[n]{\lim_{x \to a} f(x)}$
again, restrictions on *f* for *n* = even or odd.

Example, Expanding Limit Notation:

$$\lim_{x\to 2}\frac{x^3+4x^2-1}{3-2x} = \frac{\lim_{x\to 2} x^3+4x^2-1}{\lim_{x\to 2} 3-2x} = \frac{\lim_{x\to 2}(x^3) + \lim_{x\to 2}(4x^2) - \lim_{x\to 2}(1)}{\lim_{x\to 2}(3) - \lim_{x\to 2}(2x)}$$

$$= \frac{\lim_{x\to 2}(x^3) + 4\cdot\lim_{x\to 2}(x^2) - \lim_{x\to 2}(1)}{\lim_{x\to 2}(3) - 2\cdot\lim_{x\to 2}(x)} = \frac{(2)^3+4(2)^2-(1)}{(3)-2(2)} = \frac{8+16-1}{3-4}$$

$$= \frac{23}{-1} = -23$$

From this example, note that when you have a polynomial (as we had one in the numerator and another one in the denominator) or a rational function (i.e. poly / poly) the limit can be found by direct substitution.

From above:

$$\lim_{x\to 2}\frac{x^3+4x^2-1}{3-2x} = \frac{(2)^3+4(2)^2-1}{3-2(2)} = \frac{8+16-1}{3-4} = -23 \quad \text{(as before)}$$

Other functions that behave like this are:

Trigonometric Functions

Example:

$$\lim_{x\to\frac{\pi}{2}} \sin(x) = \sin\left(\frac{\pi}{2}\right) = 1$$

And, generally:

$$\lim_{\theta\to a}\sin(\theta) = \sin(a) \quad , \quad \lim_{\theta\to a}\cos(\theta) = \cos(a)$$

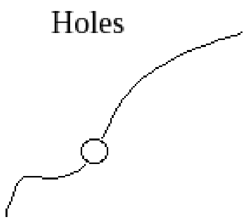
In fact, any function that is continuous at *a* has the direct substitution property – we saw that last time!

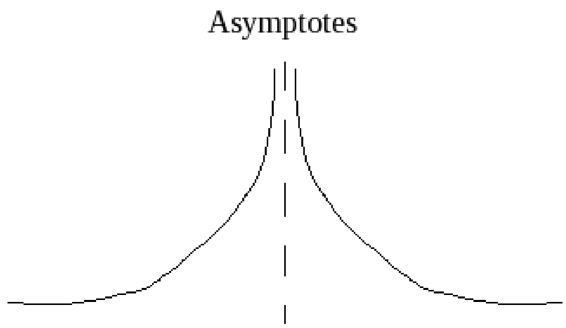
Formally:

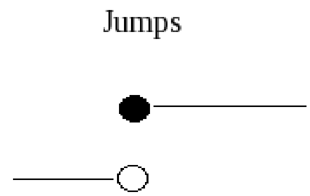
If *f(x)* is continuous at *a*:

$$\lim_{x\to a} f(x) = f(a)$$

But, what happens when we are discontinuous? What kinds of discontinuities can you think of?







A full limit exists here

DNE Here

only one sided limits here

How do we handle these and know what we have?

Example:

$$\lim_{x\to -4}\frac{x^2+5x+4}{x^2+3x-4} = \frac{(-4)^2+5(-4)+4}{(-4)^2+3(-4)-4} = \frac{16-20+4}{16-12-4} = \frac{0}{0}$$ (bad, but this does <u>not</u> equal 0!)

I see a rational function so I think substitution. However, you cannot say anything about the limit.

So, factor and try again.

$$\lim_{x\to -4}\frac{x^2+5x+4}{x^2+3x-4} = \lim_{x\to -4}\frac{\cancel{(x+4)}(x+1)}{\cancel{(x+4)}(x-1)} = \lim_{x\to -4}\frac{(x+1)}{(x-1)} = \frac{-4+1}{-4-1} = \frac{-3}{-5} = \frac{3}{5}$$

A cancellation means you have a hole at (x – constant) = 0, whatever that constant may be.
Also, you can always graph the function to find the limit visually.

Rationalization:
Example:

$$\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}$$ (multiply by the conjugate pair of radical part)

$$\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}\cdot\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3} = \lim_{x\to 7}\frac{(x+2)-9}{(x-7)(\sqrt{x+2}+3)} = \lim_{x\to 7}\frac{\cancel{(x-7)}}{\cancel{(x-7)}(\sqrt{x+2}+3)}$$

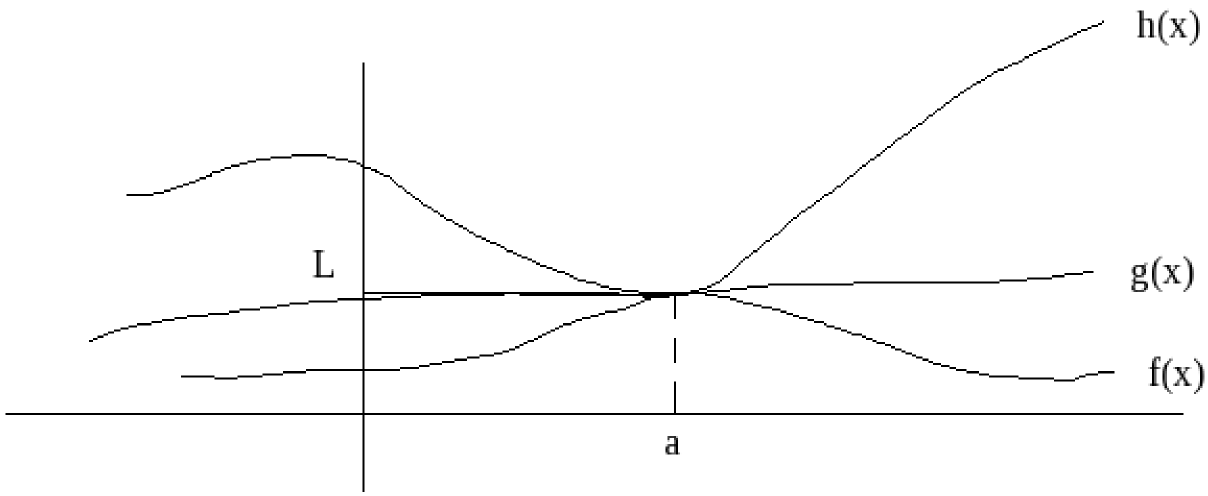
$$= \lim_{x\to 7}\frac{1}{\sqrt{x+2}+3} = \frac{1}{\sqrt{7+2}+3} = \frac{1}{3+3} = \frac{1}{6}$$

Other Ideas

<u>Squeeze Theorem</u> (also know nas sandwich theorem)
If $f(x)\le g(x)\le h(x)$ when x is near a (except possibly at a) and,

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \quad \text{then,} \quad \lim_{x\to a} g(x) = L$$

Graphically:



When would I need this? This comes up with trigonometric functions quite a bit.

Example:

$$\lim_{x \to 0} x^2 \cdot \sin\left(\frac{1}{x}\right) = ?$$

We first see that $\lim_{x \to 0} x^2 \cdot \sin\left(\frac{1}{x}\right) = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

But that doesn't help us because of that 0 in the denominator.

So, our only other option at this point is to try bounding with the squeeze theorem.

1st, we know: $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ because that's the range of sine.

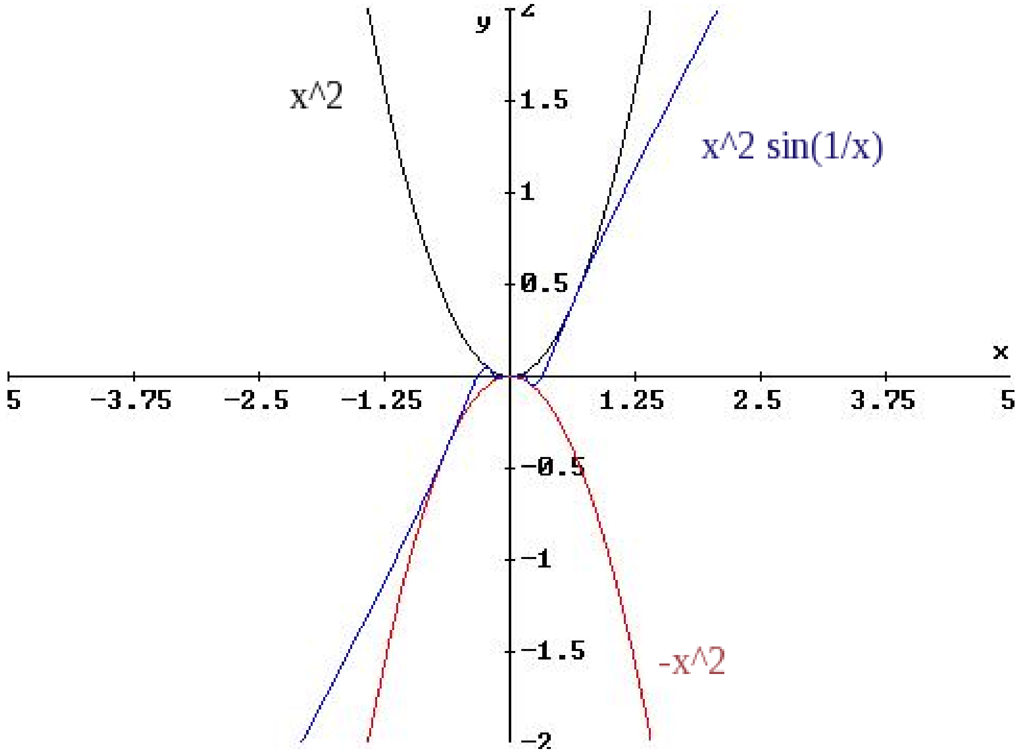
Now, multiply through by x^2 to get $-x^2 \leq x^2 \cdot \sin\left(\frac{1}{x}\right) \leq x^2$.

Evaluate the limits:

$\lim_{x \to 0} -x^2 = 0$ and $\lim_{x \to 0} +x^2 = 0$, so by squeeze theorem,

$$\lim_{x \to 0} x^2 \cdot \sin\left(\frac{1}{x}\right) = 0$$

Graphically:



Finally, limits by invoking other limits.

Recall in Section 1.3: $\lim_{x \to 0} \frac{\sin x}{x} = 1$ which we found numerically.

Example:

$$\lim_{x \to 0} \frac{\sin 14x}{3x} = ?$$

If I use direct substitution, I get a zero in the denominator. Not good. So, I have to do something else.

What I want to do is force the argument of sin to look the same as the denominator.

So: $\lim_{x \to 0} \frac{\sin 14x}{3x} = \lim_{x \to 0} \left(\frac{14}{14}\right) \cdot \frac{\sin 14x}{3x}$, or simply multiply by 1.

Rewrite as: $\lim_{x \to 0} \frac{14}{3} \cdot \frac{\sin 14x}{14x} = \frac{14}{3} \cdot \lim_{x \to 0} \frac{\sin 14x}{14x}$

Now notice: as *x* goes towards *0*, what is happening to *14x*? It goes even faster.
Now, let $y = 14x$

$$\frac{14}{3} \cdot \lim_{y \to 0} \frac{\sin y}{y} = \frac{14}{3}(1) = \frac{14}{3}$$

Example:

$$\lim_{t\to 0}\frac{\tan(4t)}{\sin(3t)} = \lim_{t\to 0}\left(\frac{\sin(4t)}{\cos(4t)}\cdot\frac{1}{\sin(3t)}\cdot\frac{t}{t}\right) = \lim_{t\to 0}\left(\frac{\sin(4t)}{t}\cdot\frac{1}{\cos(4t)}\cdot\frac{t}{\sin(3t)}\right) =$$

note: introduce t / t to force the desired form.

$$= \lim_{t\to 0}\left(\frac{4\cdot\sin(4t)}{4t}\cdot\frac{1}{\cos(4t)}\cdot\frac{3t}{3\cdot\sin(3t)}\right)$$

Force the coefficients by using multiplication by "1"

$$= 4\cdot\lim_{t\to 0}\frac{\sin(4t)}{4t}\cdot\lim_{t\to 0}\frac{1}{\cos(4t)}\cdot\frac{1}{3}\cdot\lim_{t\to 0}\frac{3t}{\sin(4t)} = 4(1)\cdot\frac{(1)}{(1)}\cdot\frac{1}{3}(1) = \frac{4}{3}$$

Example:

$$\lim_{x\to 0}\frac{x}{\sin x} = ?$$

Recall: $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Use algebra to force structure:

$$\lim_{x\to 0}\frac{x}{\sin x} = \lim_{x\to 0}\frac{x}{\sin x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{\frac{\sin x}{x}} = \frac{\lim_{x\to 0}1}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{1}{1} = 1$$

Example:

$$\lim_{x\to 0}\frac{\cos(x)-1}{x} = ?$$

$$\lim_{x\to 0}\frac{\cos(x)-1}{x} = \lim_{x\to 0}\frac{\cos(x)-1}{x}\cdot\frac{(\cos(x)+1)}{(\cos(x)+1)}$$ kind of like rationalization

This time multiplying by $\frac{1/x}{1/x}$ will not help us because there is no sine.

$$= \lim_{x\to 0}\frac{\cos^2(x)-1}{x(\cos(x)+1)}$$

Recall: $\sin^2 x+\cos^2 x = 1 \Rightarrow \sin^2 x = 1-\cos^2 x$

$$= \lim_{x\to 0}\frac{-\sin^2 x}{x(\cos(x)+1)} = \lim_{x\to 0}\frac{\sin(x)}{x}\cdot\frac{-\sin(x)}{\cos(x)+1} = \left(\lim_{x\to 0}\frac{\sin(x)}{x}\right)\cdot\left(\lim_{x\to 0}\frac{-\sin(x)}{\cos(x)+1}\right)$$

$$= (1)\left(\frac{0}{1+1}\right) = (1)(0) = 0$$