Section 1.3 The Limit of a Function

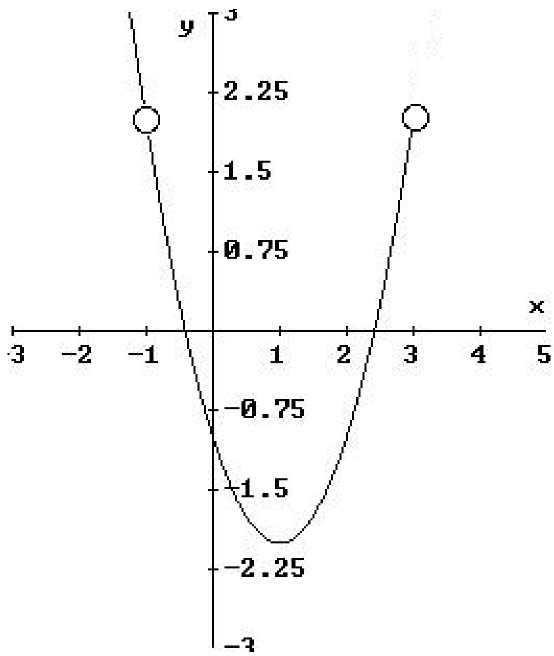
(pg. 25) **Definition**

We write $\lim_{x \to a} f(x) = L$ and say,

"The limit of *f(x)*, as *x* approaches a equals *L*"

If we can make the values of *f(x)* arbitrarily close to *L* (as close to *L* as we like) by taking *x* to be sufficiently close to *a* (on either side of *a*) but not equal to *a*.

Visual Idea:



Suppose: $f(x) = x^2 - 2x - 1$

And, let's further suppose: $x \in (-\infty, -1) \cup (-1, 3)$

Notice that how we have preset the domain, we have holes in our graph and, in fact, no graph once we pass x = 3.

Now, let's talk about this graph with reference to our definition, and the notion:

$$\lim_{x \to a} f(x) = L$$

Let's pick values for *a* and see what happens.

***a* = 0**

$\lim_{x \to 0} (x^2 - 2x - 1) = ?$ (-1)

While looking at the plot, trace the values of the graph from the left and right of the point *x = 0*. Notice that the y-values must come to the same point as I approach *x = 0* from both the right and left.

Since, *f(x)* is continuous there, we can substitute *x = 0* into *f(x)* to find the limit, *L*.

***a* = -1**

$\lim_{x \to -1} (x^2 - 2x - 1) = ?$ (2)

Even though I do not have a y-value (it's undefined – a hole), I can still have a limit, because the same y-value is being approached as *x* gets close to *x = -1* from BOTH the right and left. Being undefined is inconsequential.

***a* = 3**

$\lim_{x \to 3} (x^2 - 2x - 1) = ?$ DNE

This time the limit does not exist. Why? Because we must be able to approach *x = 3* from BOTH the right and left. This time, we can only come from the left. There is no right. We will learn more about one-sided limits later in this section.

It is possible to evaluate limits without ever having a graph. You can do it algebraically or numerically, by making tables.

pg. 33, #11: Algebraically: $\lim_{x \to 2} \frac{x^2 - 2x}{x^2 - x - 2} = \lim_{x \to 2} \frac{x(x-2)}{(x-2)(x+1)} = \lim_{x \to 2} \frac{x}{x+1} = \frac{2}{3}$

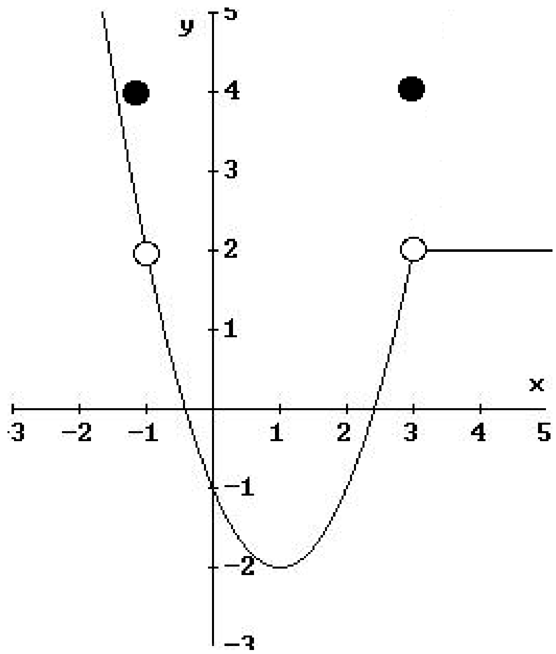
Numerically:

x	$\frac{x^2 - 2x}{x^2 - x - 2}$
1.9	0.655172
1.99	0.665552
1.999	0.666556
2	2/3
2.01	0.667774
2.1	0.677419

Notice: the function above is not defined at x = 2, but it still has a limit at x = 2.

Example: Let's go back to: $f(x) = x^2 - 2x - 1$ and redefine it slightly as a piece-ways function.

$$g(x) = \begin{cases} x^2 - 2x - 1 & x \in (-\infty, -1) \cup (-1, 3) \\ 4 & x = 3, \text{ and}, \ x = -1 \\ 2 & x \in (3, \infty) \end{cases}$$



$\lim_{x \to 3} g(x) = 2$ Even though *g(3)* = 4.

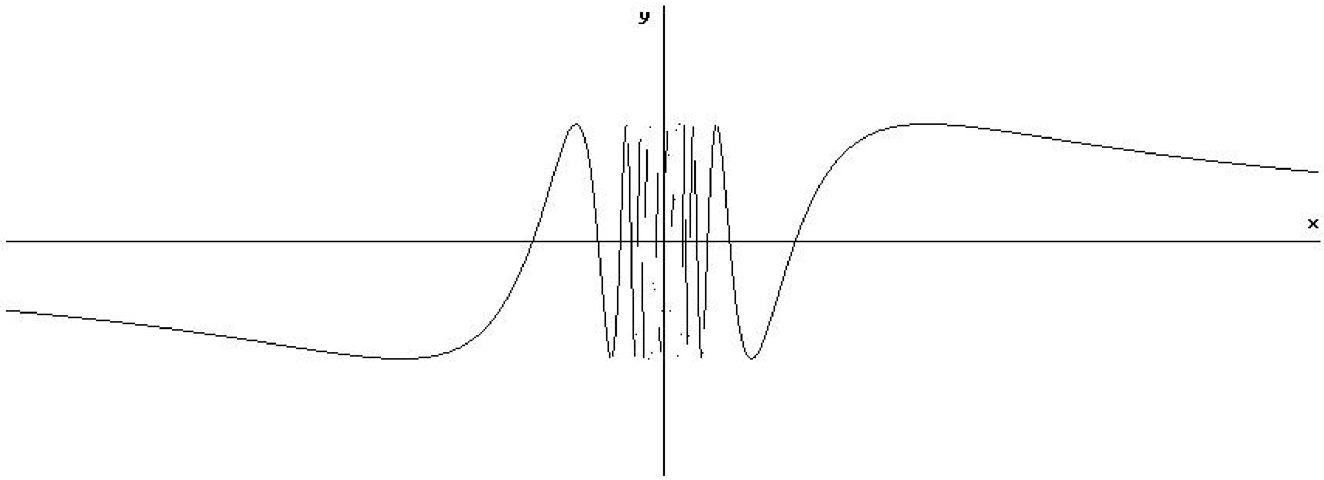
Example: $\lim_{x \to 0} \frac{\sin x}{x} = 1$ **We need to memorize this one as a building block.**

Numerically:

x	$\frac{\sin x}{x}$
0.1	0.998334
0.01	0.999983
0.001	0.999999
0	1
- 0.001	0.999999
- 0.01	0.999983
- 0.1	0.998334

Example: $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$ Does Not Exist.

Graphically:



Why? Choose $x = \frac{1}{n}$ $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = ?$

$x = \frac{1}{100}$ $\rightarrow$ 0

$x = \frac{1}{1000}$ $\rightarrow$ 0

$x = \frac{1}{10000}$ $\rightarrow$ 0

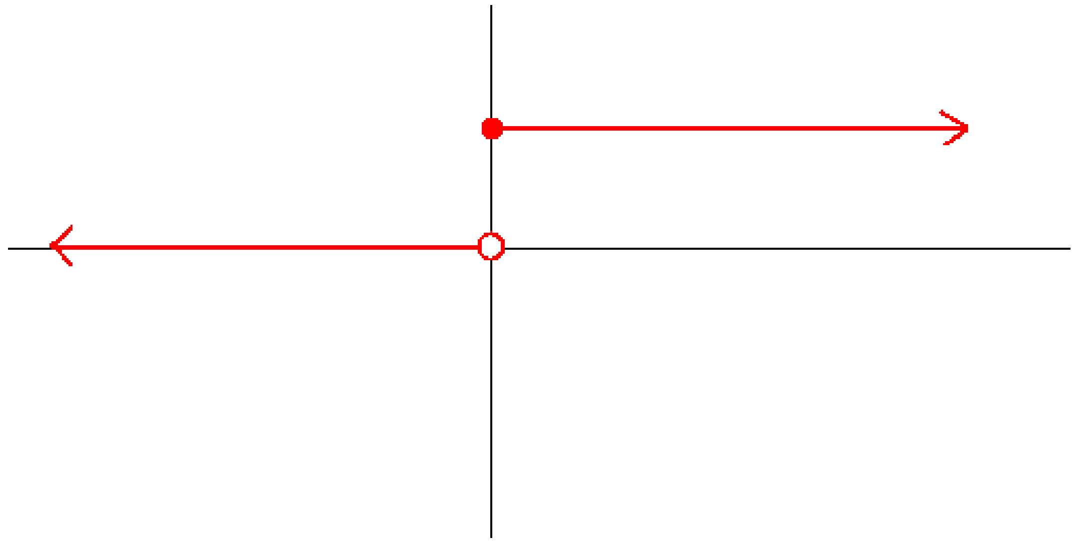
But,

$x =$ others $\rightarrow$ 1

$x = ...$ $\rightarrow$ -1

$x = ...$ $\rightarrow$ 1

One Sided Limits



Called the Heaviside Function: $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 1 \end{cases}$

$\lim_{x \to 0^+} f(x) = 1$
"From the Right"

$\lim_{x \to 0^-} f(x) = 0$
"From the Left"

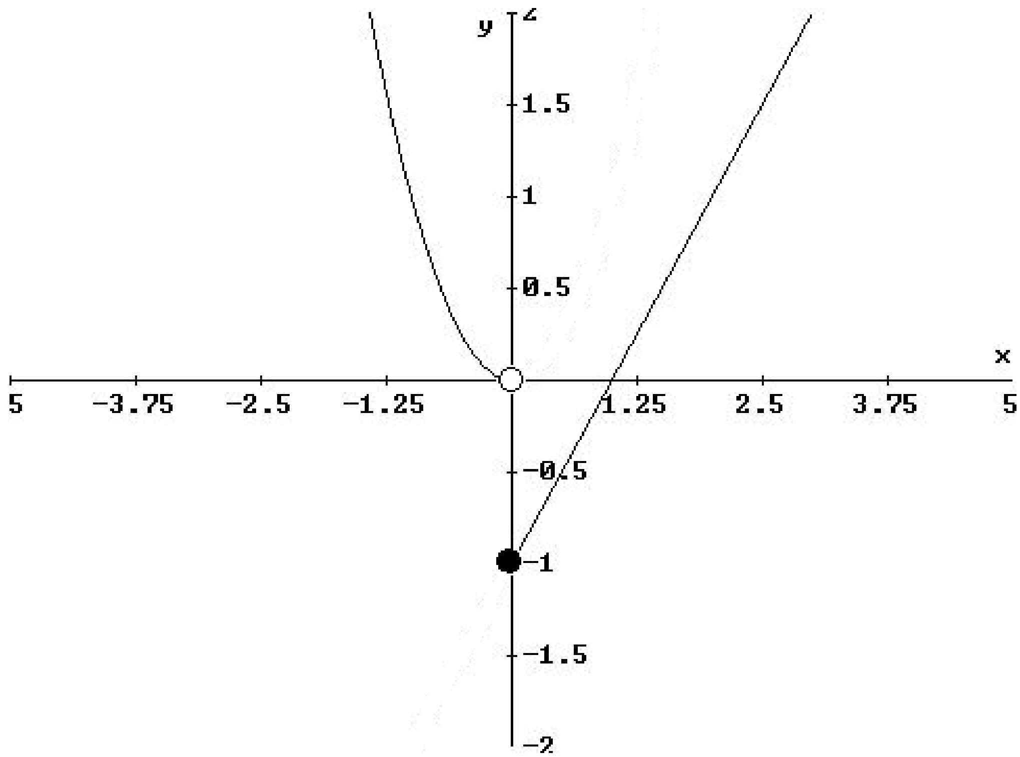
General Notation: $\lim_{x \to a^-} f(x) = L$ Left Hand Limit

The limit as x approaches *a* from the left. (Note the small – sign behind the *a*)

$\lim_{x \to a^+} f(x) = L$ Right Hand Limit

The limit as x approaches *a* from the right. (Note the small + sign behind the *a*)

Example:



$\lim_{x \to 0^-} g(x) = 0$ $\lim_{x \to 0^+} g(x) = -1$ $\lim_{x \to 0} g(x) = \text{DNE}$