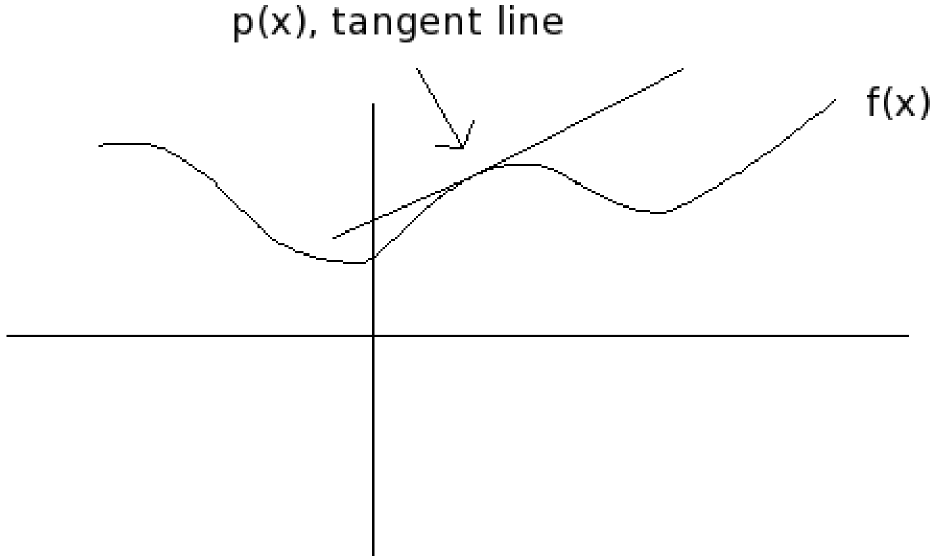
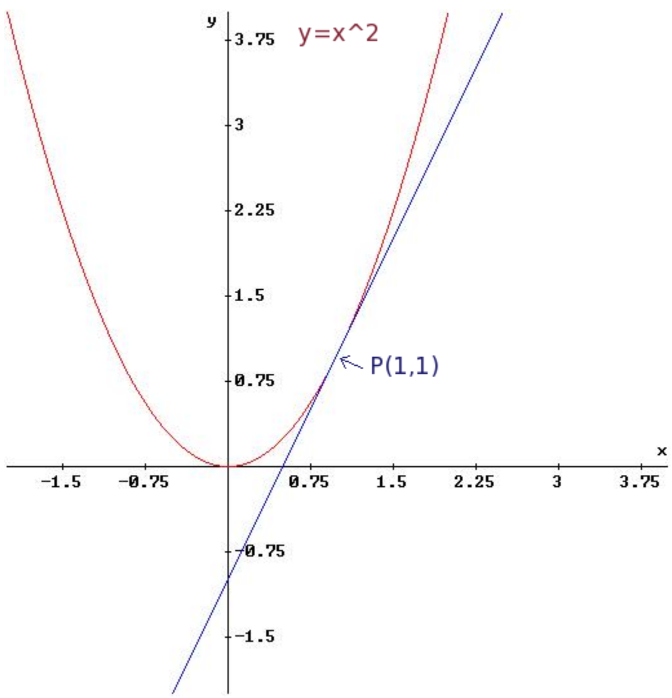
Section 2.1 Derivatives and Rates of Change



How do we find an equation for any line? Up until now, we need slope and one point or we need two points. (So, how can we find slope...)

What if we only know 1 point?



Say, we want to know the tangent line at the point (1, 1)

How do we guess it?

1. Choose a nearby point $Q\left(x, f(x) = x^2\right)$ or $Q(x, x^2)$

2. Figure slope between *P* and *Q*: $\dfrac{\Delta y}{\Delta x} = \dfrac{x^2-1}{x-1}$

3. Then let's slide *Q* until it is on top of *P*.

   Notation: $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = \lim\limits_{x\to 1} \dfrac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = 2$

4. Now:
$$y - y_1 = m(x - x_1)$$
$$y - 1 = 2(x-1)$$
$$y = 2x - 1$$

If we want to generalize this procedure.

$P: \left(a, f(a)\right)$

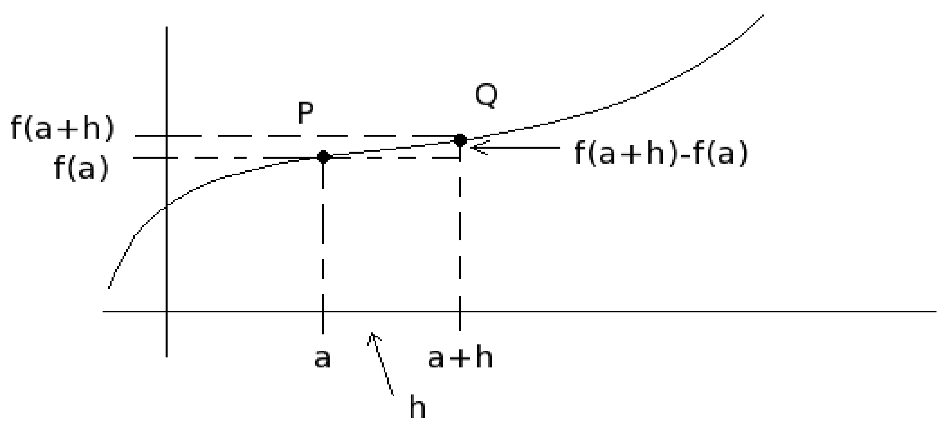
$Q: \left(x, f(x)\right)$

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x)-f(a)}{x-a}$$

Then, letting *Q* approach *P*.

$m = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$, this is also the slope of the tangent line at the point, *P*.

You might also see this written another way.



General: $m = \dfrac{f(a+h)-f(a)}{a+h-a} = \dfrac{f(a+h)-f(a)}{h}$

Then, letting *Q* slide over to *P*: $m = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

How does this relate to derivatives?

Definition: The derivative of a function, *f*, at a number *a*, which we will denote as *f ' (a)* is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{if this limit exists.}$$

So, the derivative of a function at any point, *a*, is just the slope of the tangent line at that point!

Notation Change: What if I want to express this in terms of *x* instead of *h*?

Let: $x = a + h$ then: $h = x - a$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{(x-a) \to 0} \frac{f(x) - f(a)}{x-a} = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

Calculation:

Example (p. 82, #26): Find the derivative of the function $f(x) = \frac{x^2+1}{x-2}$ at the number *a*.

Apply form: $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$$f(a+h) = \frac{(a+h)^2+1}{(a+h)-2} = \frac{a^2+2\text{ah}+h^2+1}{(a-h)+2} \qquad f(a) = \frac{a^2+1}{a-2}$$

Substitute in:

$$f'(a) = \lim_{h \to 0} \frac{\frac{a^2+2\text{ah}+h^2+1}{(a-h)+2} - \frac{a^2+1}{a-2}}{h} = \lim_{h \to 0} \frac{(a^2+2\text{ah}+h^2+1)(a-2) - (a^2+1)(a+h-2)}{h(a+h-2)(a-2)}$$

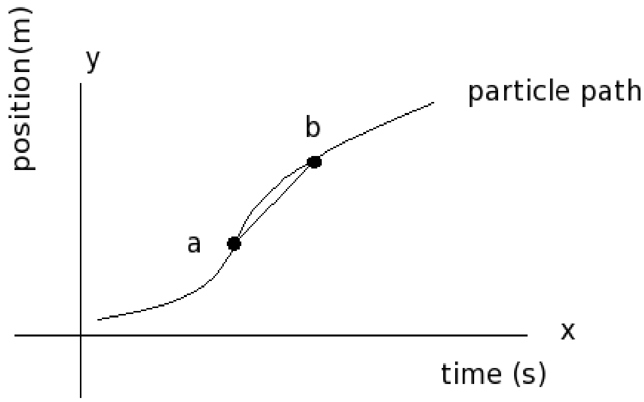
$$= \lim_{h \to 0} \frac{\cancel{a^3} + 2a^2 h + ah^2 + \cancel{a} - \cancel{2a^2} - 4\text{ah} - 2h^2 - \cancel{2} - [\cancel{a^3} + a^2 h - \cancel{2a^2} + \cancel{a} + h - \cancel{2}]}{h(a+h-2)(a-2)}$$

$$= \lim_{h \to 0} \frac{a^2 h + ah^2 - 4\text{ah} - 2h^2 - h}{h(a+h-2)(a-2)} = \lim_{h \to 0} \frac{\cancel{h}(a^2 + ah - 4\text{a} - 2\text{h} - 1)}{\cancel{h}(a+h-2)(a-2)} = \lim_{h \to 0} \frac{a^2 + ah - 4\text{a} - 2\text{h} - 1}{(a+h-2)(a-2)}$$

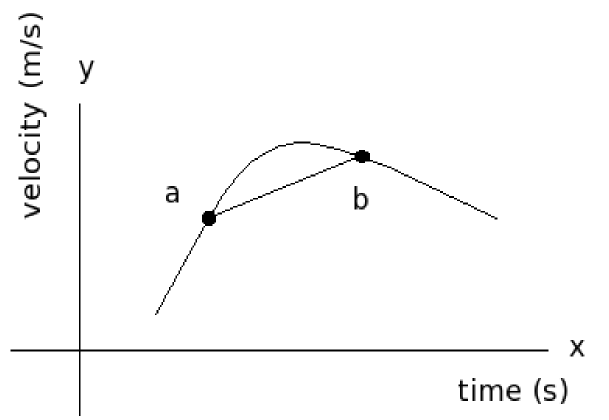
$$= \frac{a^2 + a(0) - 4\text{a} - 2(0) - 1}{(a+0-2)(a-2)} = \frac{a^2 - 4\text{a} - 1}{(a-2)(a-2)} = \boldsymbol{\frac{a^2 - 4a - 1}{(a-2)^2}}$$

Application:

Since the derivatives sort of act like instantaneous slopes, they have many engineering applications. The most popular is the relationship between position, velocity, and acceleration.



Unit of slope = $\frac{\Delta y}{\Delta x} = \frac{\text{m}}{\text{s}}$ Which is what? Velocity. So, the derivative of a position function gives us velocity.



Unit of slope = $\frac{\Delta y}{\Delta x} = \frac{m/s}{s} = \frac{m}{s^2}$ Which is what? Acceleration. So, the derivative of a velocity function gives us acceleration.

Likewise, we can determine units for any number of slope relationships.

Example: An arrow is shot upward on the moon with a velocity of 58 m/s, it's height (in m) after *t* seconds is given by:

$$s(t) = 58t - 0.83t^2$$

Find the velocity of the arrow after 1 second.

$$s'(1) = \lim_{h \to 0} \frac{58(1+h) - 0.83(1+h)^2 - [58(1) - 0.83(1)^2]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{58} + 58\text{h} - 0.83[1 + 2\text{h} + h^2] - \cancel{58} + 0.83}{h} = \lim_{h \to 0} \frac{58\text{h} - \cancel{0.83} - 1.66\text{h} - 0.83\text{h}^2 + \cancel{0.83}}{h}$$

$$s'(1) = \lim_{h \to 0} \frac{\cancel{h}(58 - 1.66 - 0.83h)}{\cancel{h}} = 58 - 1.66 - 0.83(0) = \mathbf{56.34}\ \frac{\boldsymbol{m}}{\boldsymbol{s}}$$

Definition: Instantaneous Rate of Change.

$$\lim_{x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{(another way you will see this)}$$

So, instantaneous rate of change is just a derivative $f'(a)$ of $y = f(x)$ with respect to x when $x = a$.