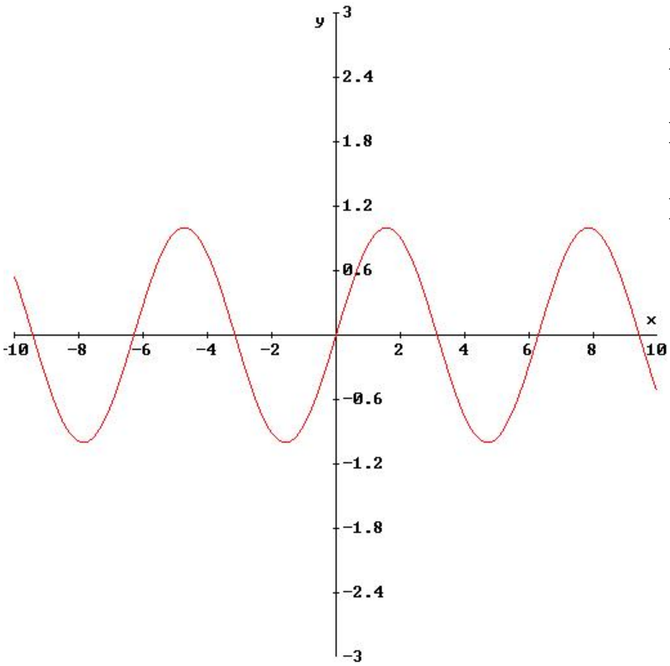
Section 3.5 Inverse Trigonometric Functions



f(x) = sin(x)
By inspection, does this function have an inverse?

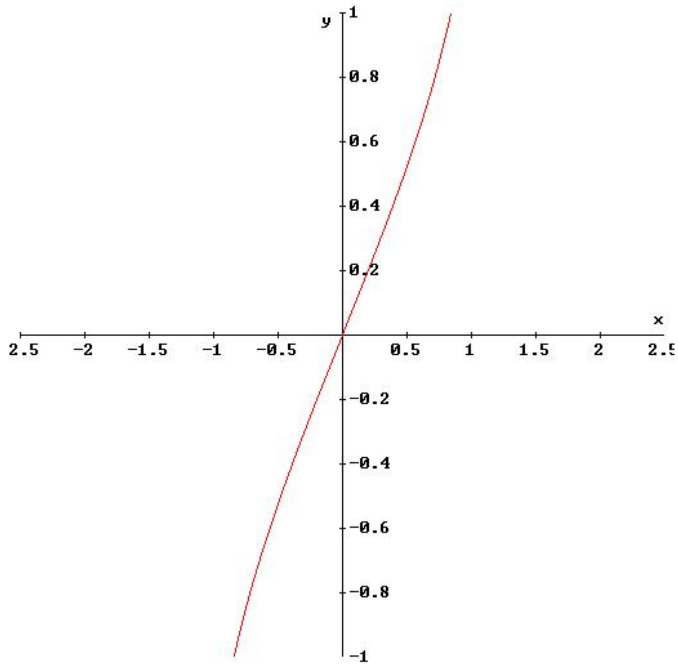
No, it fails the horizontal line test. (not one-to-one)

Hm...but we have seen things like $\sin^{-1} x$ how??

Domain Limitations

For sin(x), if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it passes the horizontal line test and is locally one-to-one.

What does $\sin^{-1} x$ look like?



So, $\sin^{-1} x = y \Longleftrightarrow \sin y = x \quad \text{if} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Domain $\sin^{-1}(x) = [-1, 1]$ Range $\sin^{-1}(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Note: $\sin^{-1} \neq \frac{1}{\sin x}$

Example: $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad \Leftarrow\Rightarrow \quad \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \underbrace{\frac{\pi}{3}}_{\text{Must be between } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)}$ Unit Circle

Notation: Sometimes instead of $\sin^{-1} x = \arcsin x$

Example: $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad \Leftarrow\Rightarrow \quad \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \neq \underbrace{\frac{2\pi}{3}}_{\text{Must be between } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)}$

Instead, always $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

Just like other inverse functions: $f\left(f^{-1}(x)\right) = x$

So: $\sin\left(\sin^{-1}(x)\right) = x$

And: $\sin^{-1}\left(\sin(x)\right) = x$

Example: Because $\sin x$ is continuous and differentiable so is $\sin^{-1} x$. How do we find its derivative?

Implicitly: $y = \sin^{-1} x \underset{\text{Hence we know}}{\Rightarrow} \sin y = x \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

The derivative will be $\frac{dy}{dx}$

Work on easier function:

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$
$$\cos y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Here's the tricky part: we know $\cos y \geq 0$ since $-\frac{\pi}{2} \leq y$

Recall: $\sin^2 y + \cos^2 y = 1$
$\cos^2 y = 1 - \sin^2 y$

Take the positive one: $\cos y = +\sqrt{1-\sin^2 y}$

So: $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\sin^2 y}}$

But, what is sin y? x (from above)

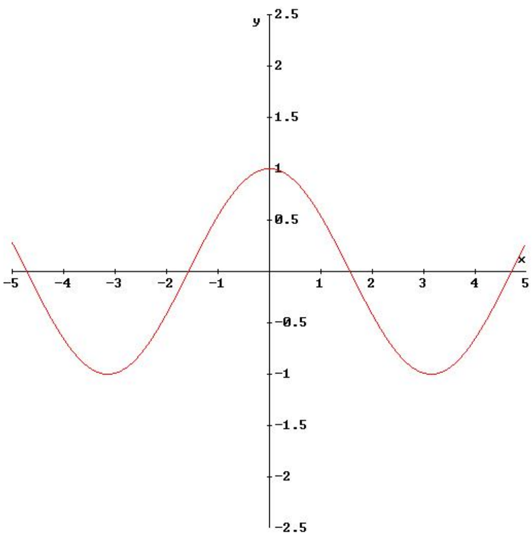
So: $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

And $\dfrac{d}{dx}\left[\sin^{-1}(x)\right] = \dfrac{1}{\sqrt{1-x^2}}$ provided $-1 < x < 1$

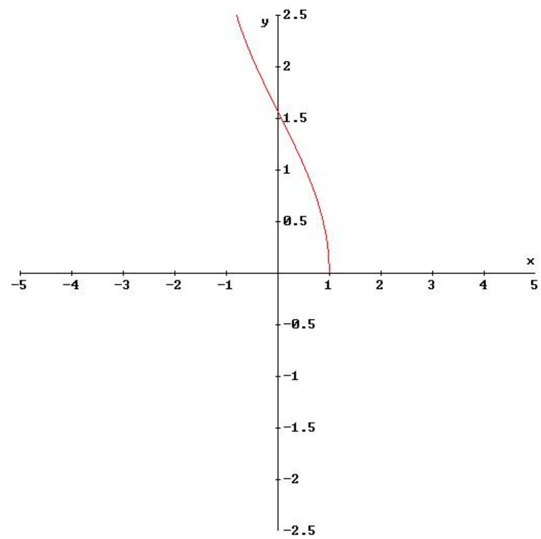
Example: Work $\cos^{-1} x$ on your own or through this example.

Limit the domain to $x \in [0, \pi]$ (now the function locally is one-to-one)

So, $y = \cos^{-1} x \quad \Rightarrow \quad \cos y = x \quad 0 \le y \le \pi$



$\cos x$



$\cos^{-1} x$

1. Domain $\cos^{-1} x$: [-1, 1]
2. Range $\cos^{-1} x$: [0, π]

Note: $\cos^{-1} x \neq \dfrac{1}{\cos x}$

Find derivative:

Let $y = \cos^{-1} x \quad \Leftarrow\Rightarrow \quad x = \cos y$

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\cos^{-1}(x)\right)$

Start with: $x = \cos y$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\cos y)$$
$$1 = -\sin y \frac{dy}{dx}$$
$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

Tricky part, sign of sine when input $0 \le y \le \pi$

$$\sin y > 0$$
$$\sin^2 y + \cos^2 y = 1$$
$$\sin^2 y = 1 - \cos^2 y$$
$$\sin y = +\sqrt{1 - \cos^2 y}$$
$$\sin y = +\sqrt{1 - x^2}$$

So: $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$ provided $-1 < x < 1$

Formally: $\frac{d}{dx}\left[\cos^{-1}(x)\right] = -\frac{1}{\sqrt{1-x^2}}$

Example: Evaluate $\cos^{-1}\left(\frac{-\sqrt{2}}{2}\right) = \frac{3\pi}{4}$ (must be between [0, π])

Example: Work $\tan^{-1} x$ on your own or through this example.

Limit the domain

$$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

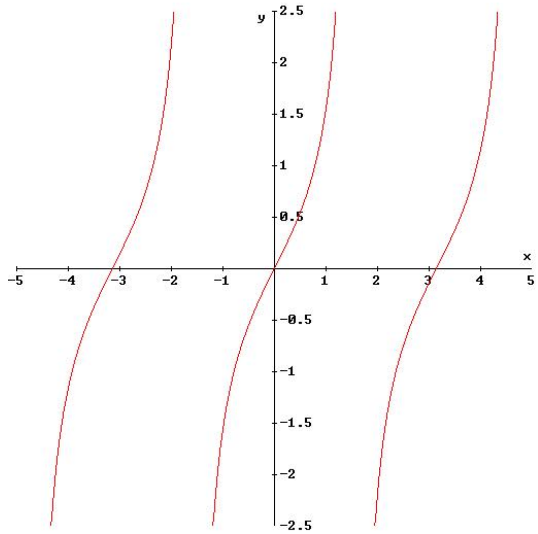
Domain: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Range: $(-\infty, \infty)$

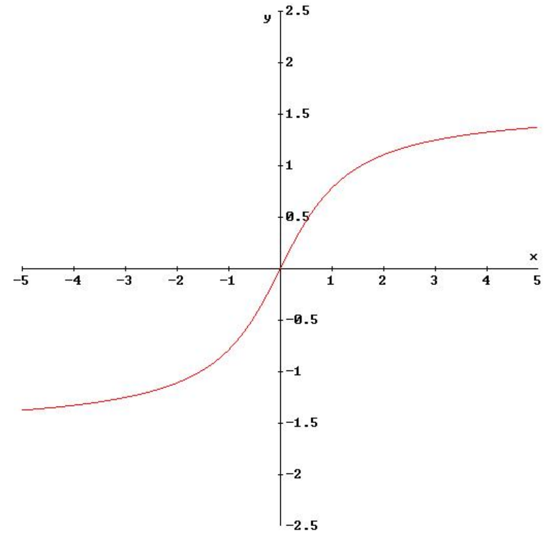
So, $y = \tan^{-1} x \quad \Leftarrow\Rightarrow \quad \tan y = x \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Domain $\tan^{-1} x$: $(-\infty, \infty)$

Range $\tan^{-1} x$: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$\tan x$



$\tan^{-1} x$

Find derivative

$$y = \tan^{-1} x \quad \Leftarrow\Rightarrow \quad x = \tan y$$

Let $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\tan^{-1}(x)\right)$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$$

$$1 = \sec^2 y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$1+\tan^2 y = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

So: $\dfrac{d}{dx}\left[\tan^{-1}(x)\right] = \dfrac{1}{1+x^2}$

Example: Find an exact value for:

1. $\sin^{-1}\left(\dfrac{1}{2}\right) = ?$

How to think through:

What angle between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ (i.e. quadrant 4 or 1) has $\sin x = \dfrac{1}{2}$?

Quadrant 1, $\dfrac{\pi}{6}$ family $\Rightarrow$ $\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$

2. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = ?$

Where is $\sin x = -\frac{\sqrt{3}}{2}$? Quadrant 4, $\frac{\pi}{3}$ family $\Rightarrow$ $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

3. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = ?$

What angle between 0 and π (i.e. quadrant 1 or 2) has $\cos x = \frac{\sqrt{2}}{2}$?

Quadrant 1, $\frac{\pi}{4}$ family $\Rightarrow$ $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

4. $\cos^{-1}\left(-\frac{1}{2}\right) = ?$

Where is $\cos x = -\frac{1}{2}$? quadrant 2, $\frac{\pi}{3}$ family $\Rightarrow$ $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

5. $\tan^{-1}(-1) = ?$

What angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (i.e. quadrant 4 or 1) has $\tan x = -1$?

Quadrant 4, $\frac{\pi}{4}$ family $\Rightarrow$ $\tan^{-1}(-1) = -\frac{\pi}{4}$

6. $\tan^{-1}\left(\sqrt{3}\right) = ?$

$$\tan x = \sqrt{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \quad \begin{matrix} \leftarrow \sin x \\ \leftarrow \cos x \end{matrix}$$

Example: Simplify the expression:

$\cos\left(\tan^{-1}(x)\right)$

Let $y = \tan^{-1} x$

Then $\tan y = x$ $\quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

So, we are trying to find $\cos y$

Relationships known: $\sec^2 y = 1+\tan^2 y$

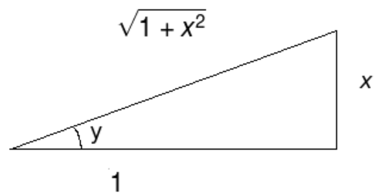
$$\sec^2 y = 1+x^2$$

$$\sec y = \sqrt{1+x^2}$$

So $\cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1+x^2}}$

Hence: $\cos\left(\tan^{-1} x\right) = \cos y = \frac{1}{\sqrt{1+x^2}}$

Another way (pictorially):



$$\tan y = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{1} = x$$

So $\cos y = \frac{1}{\sqrt{1+x^2}}$

Example: Simplify the expression:

$$\sin\left(\tan^{-1}(x)\right)$$

From picture (above) $\sin\left(\tan^{-1}(x)\right) = \frac{x}{\sqrt{1+x^2}}$

Example: Find derivatives:

$$y = \tan^{-1}\sqrt{x}$$

Recall: $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

$$y' = \frac{d}{dx}\left(\tan^{-1}\sqrt{x}\right)$$

$$= \frac{d}{dx}\left[\tan^{-1}\left(x^{\frac{1}{2}}\right)\right]$$

Apply Chain Rule

$$= \frac{1}{1+\left(x^{\frac{1}{2}}\right)^2} \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= \frac{1}{1+x}\left(\frac{1}{2}\right)x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}(1+x)}$$

$y = \sin^{-1}(2\text{x}+1)$

Recall: $\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}$

$$y' = \frac{d}{dx}\left(\sin^{-1}(2\text{x}+1)\right)$$

Apply Chain Rule

$$= \frac{1}{\sqrt{1-(2\text{x}+1)^2}} \frac{d}{dx}(2\text{x}+1)$$

$$= \frac{2}{\sqrt{1-(2\text{x}+1)^2}}$$

$H(x) = (1+x^2)\arctan x$

$$H'(x) = \underbrace{\left(1+x^2\right)\frac{d}{dx}\left(\tan^{-1}(x)\right) + \left(\tan^{-1}x\right)\frac{d}{dx}\left(1+x^2\right)}_{\text{Product Rule}}$$

$$= \cancel{(1+x^2)} \cdot \frac{1}{\cancel{1+x^2}} + \left(\tan^{-1}x\right)(2\text{x})$$

$$= 1+2\text{x}\tan^{-1}x$$

$$= 1+2\text{x}\ \text{arctan}\ x$$

Example: Find derivative and domain of $g(x) = \cos^{-1}(3-2\text{x})$

Domain: $\cos^{-1}x$: $x \in [-1,\ 1]$

So Domain: $\cos^{-1}(3-2\text{x}) \Rightarrow (3-2\text{x}) \in [-1,\ ]1$

$$-1 \le 3-2\text{x} \le 1$$
$$-4 \le -2\text{x} \le -2$$
$$2 \ge x \ge 1$$

Domain: $x \in [1,\ 2]$
Range: $[0,\ \pi]$

Derivative:

$$g'(x) = \frac{-1}{\sqrt{1-(3-3x)^2}} \frac{d}{dx}(3-2x)$$
$$= \frac{(-1)(-2)}{\sqrt{1-(3-3x)^2}}$$
$$= \frac{2}{\sqrt{1-(3-3x)^2}}$$

Domain:

$$1-(3-2x)^2 \neq 0$$
$$(3-2x)^2 \neq 1$$
$$3-2x \neq 1$$
$$-2x \neq -2$$
$$x \neq 1 \Rightarrow x \in [1, 2]$$