

Section 3.1 Exponential Functions

Exponential Functions have the general form $f(x) = a^x$ where a is a positive constant.

Algebra Review:

So, if $x = n$ some positive integer:

1. then $a^x = a^n = \underbrace{a \cdot a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$

2. Remember, too, that $a^0 = 1$

3. and, $a^{-n} = \frac{1}{a^n}$

4. x might also be a rational number (i.e. fraction) like $\frac{p}{q}$ ($q > 0$) where p and q are integers

and

$$a^{\frac{p}{q}} = \sqrt[q]{a^p} \text{ which you can also always write as } (\sqrt[q]{a})^p$$

But, what does it mean if x happens to be irrational?

e.g. What is a^π ? this gets a little sticky...

Suppose I have $f(x) = 5^x$ and I am interested in $5^{\sqrt{5}}$

$\sqrt{5}$ is irrational

If we mash $\sqrt{5}$ into a calculator: $\sqrt{5} \approx 2.23606798$ -- an approximation.

Notice Then:

$$\begin{aligned} \frac{22}{10} = 2.2 < \sqrt{5} < 2.3 = \frac{23}{10} \\ \frac{223}{100} = 2.23 < \sqrt{5} < 2.24 = \frac{224}{100} \\ \frac{2,236}{1,000} = 2.2236 < \sqrt{5} < 2.237 = \frac{2,237}{1,000} \\ \frac{22,360}{10,000} = \underbrace{2.22360}_{\text{rational}} < \underbrace{\sqrt{5}}_{\text{irrational}} < \underbrace{2.2361}_{\text{rational}} = \frac{22,361}{10,000} \end{aligned}$$

And, I can keep going & going & going...

I can literally go on forever if I wanted to. But, notice that if we think of the rational numbers above as some kind of sequence:

$$2.2, 2.23, 2.236, \dots, \sqrt{5}, \dots, 2.2361, 2.237, 2.24, 2.3$$

The $\sqrt{5}$ acts like the upper bound to the left hand side and the lower bound to the right hand

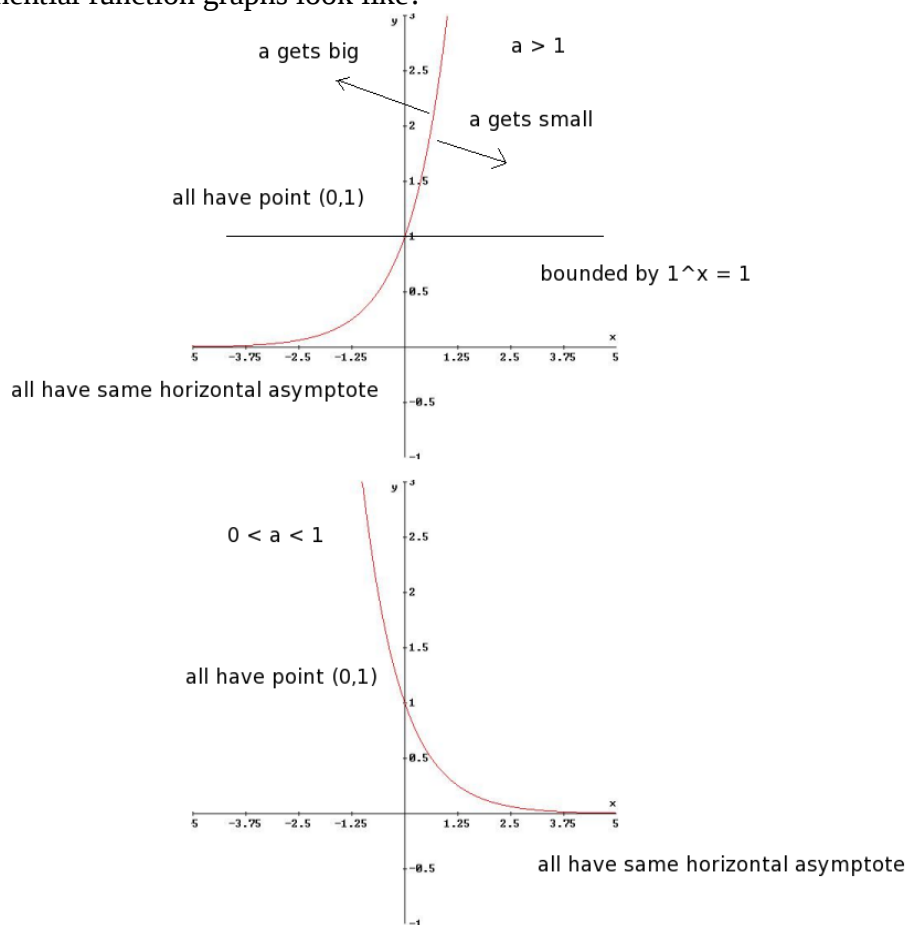
side.

What does that make $\sqrt{5}$? Some kind of limit. Aha...

So, what an exponential function really does for irrational (as well as any other) values is this:

$$a^x = \lim_{r \rightarrow x} a^r \text{ where } r \text{ is a rational number (like the ones in the sequence approximating } \sqrt{5} \text{)}$$

What do exponential function graphs look like?



More Review: (Laws of Exponents)

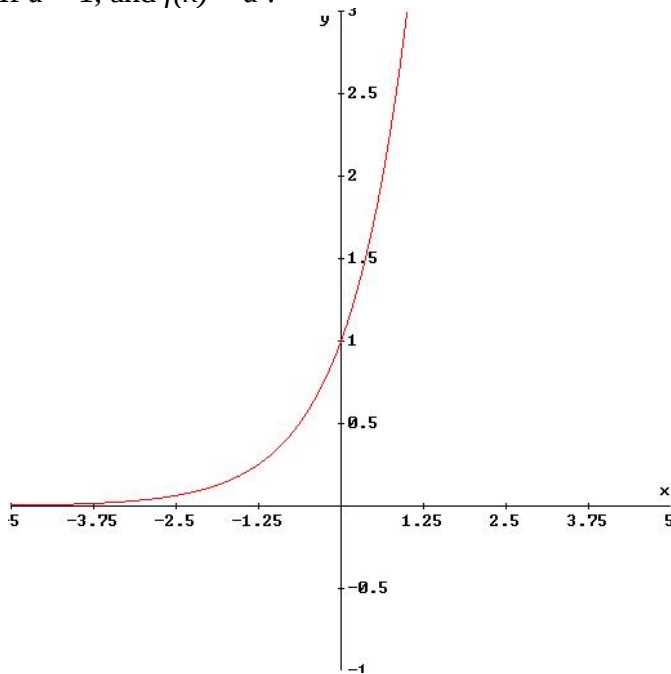
Theorem: If $a > 0$, $a \neq 1$, then $f(x) = a^x$ is a continuous function with domain \mathbb{R} (all real number) and range $(0, \infty)$. In particular $a^x > 0$ for all x .

So, if $a^x > 0$ and $x, y \in \mathbb{R}$

1. $a^{x+y} = a^x \cdot a^y$
2. $a^{x-y} = \frac{a^x}{a^y}$
3. $(a^x)^y = a^{x \cdot y}$
4. $(a \cdot b)^x = a^x \cdot b^x$

Now, calculus things to think about...

1. If $a > 1$, and $f(x) = a^x$:



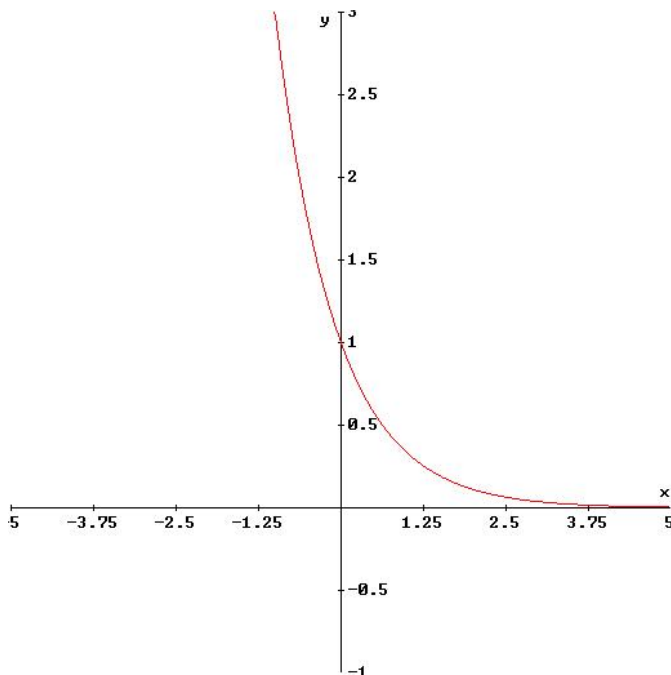
$$\lim_{x \rightarrow \infty} a^x = \infty$$

(acts like a left-hand limit)

$$\lim_{x \rightarrow -\infty} a^x = 0$$

(acts like a right-hand limit)

2. if $0 < a < 1$, and $f(x) = a^x$:



$$\lim_{x \rightarrow \infty} a^x = 0$$

(acts like a left-hand limit)

$$\lim_{x \rightarrow -\infty} a^x = \infty$$

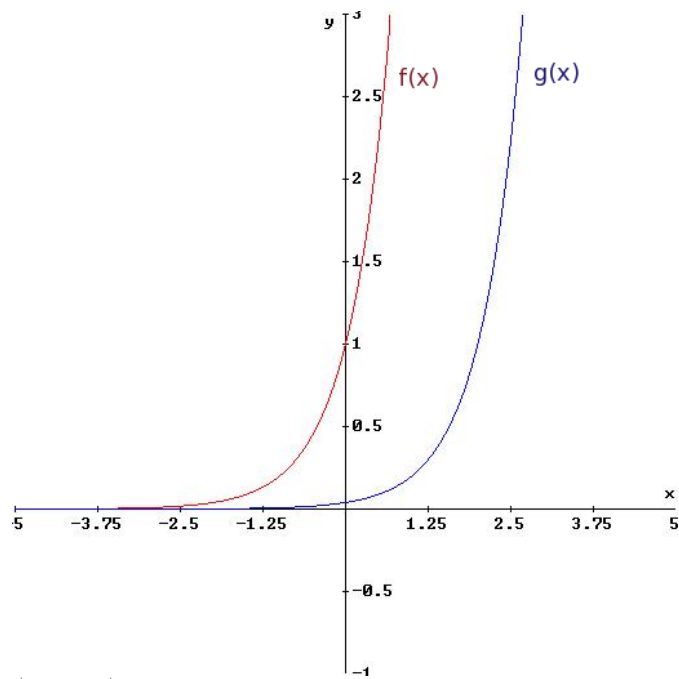
(acts like a right-hand limit)

Example: Rough Sketches (Family Rules Still Apply)

Sketch: $y = 5^{x-2}$
 $g(x) = 5^{x-2}$

parent: $y = 5^x$
 $f(x) = 5^x$
 $f(x-2) = 5^{x-2}$

So, $g(x) = f(x-2)$



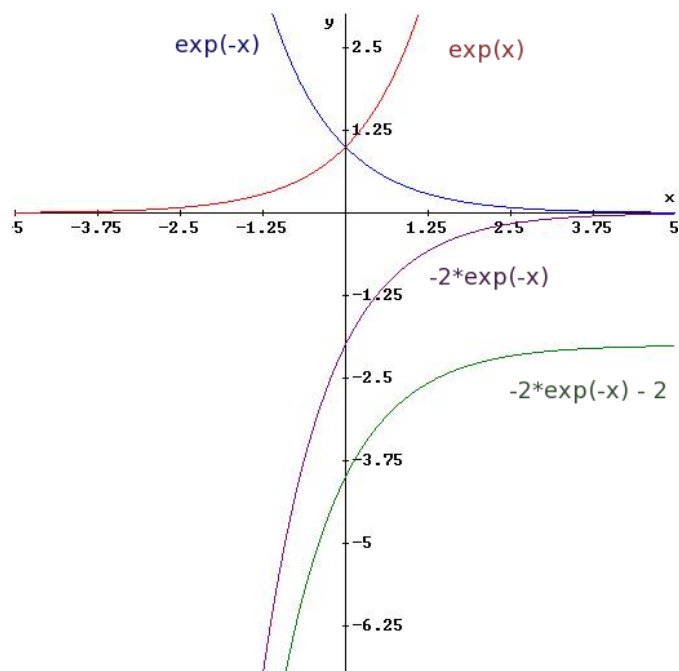
Sketch: $y = -2(1 + e^{-x})$
 $y = -2 - 2e^{-x}$ Ick!
 $g(x) = -2 - 2e^{-x}$

Parent: $f(x) = y = e^x$

$f(-x) = e^{-x}$ Flips horizontally

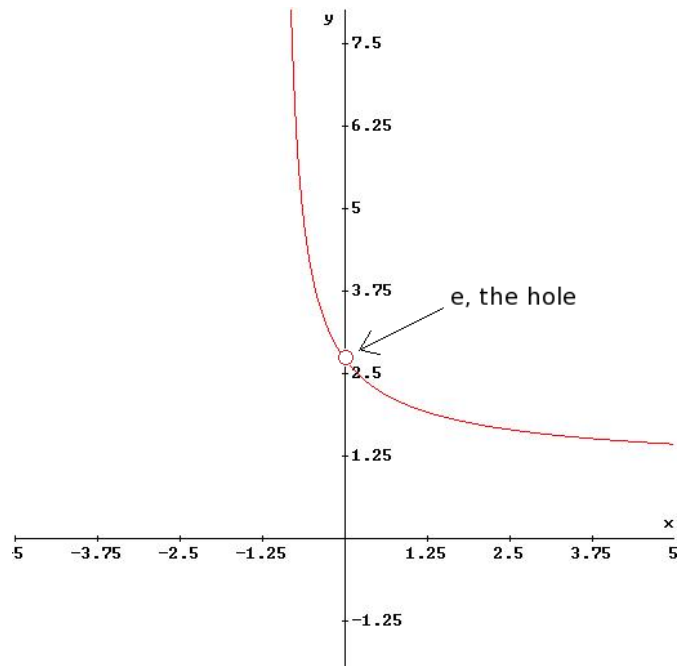
$-2f(-x) = -2e^{-x}$ Vertical scale 2, flips vertically

$g(x) = -2f(-x) - 2 = -2 - 2e^{-x}$ Vertical shift down 2 units



Definition: (shown pg. 145 in text) The value of e .

$$e = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}}$$



Calculated Numerically: ($e \approx 2.71828$)

x	$(1+x)^{\frac{1}{x}}$
0.1	2.59374246
0.01	2.70481383
0.00001	2.71826824

The hole in our graph comes from the $\frac{1}{x}$ exponent – you cannot have 0 in the denominator. But the limit as $x \rightarrow 0$ is where we get the value for e from.

Specifically $y = e^x$ is called the “natural exponential function”

* Its unique feature is that it crosses the y-axis with a slope of 1 giving us some very nice derivative properties.

Notice then too that $e > 1$ (just like $a > 1$)

So $f(x) = e^x$ has the properties that:

$$\lim_{x \rightarrow \infty} e^x = \infty \qquad \lim_{x \rightarrow -\infty} e^x = 0$$

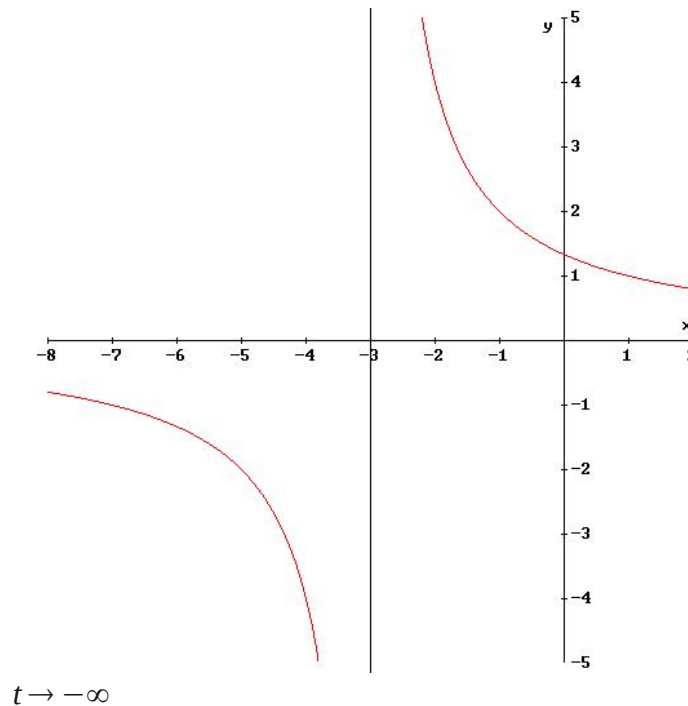
Example: Limits involving the natural exponential function

Evaluate: $\lim_{x \rightarrow \infty} e^{x^2} = \lim_{t \rightarrow \infty} e^t = \infty$
 Let $t = x^2$ as $x \rightarrow \infty, t \rightarrow \infty$

Evaluate: $\lim_{x \rightarrow \infty} e^{-x^2} = \lim_{t \rightarrow -\infty} e^t = 0$
 Let $t = -x^2$ as $x \rightarrow \infty, t \rightarrow -\infty$

Another one that is not so obvious:

Evaluate: $\lim_{x \rightarrow -3^-} e^{\frac{4}{3+x}} = \lim_{t \rightarrow -\infty} e^t = 0$ (same as before)
 Let $t = \frac{4}{3+x}$ as $x \rightarrow -3^-, t \rightarrow ?$ Graph it to find out:



Evaluate: $\lim_{x \rightarrow \infty} (e^{-3x} \cdot \sin x)$

We know: $-1 \leq \sin x \leq 1$ and $e^{-3x} > 0$

Setup squeeze theorem:

$$(-1)e^{-3x} \leq e^{-3x} \cdot \sin x \leq (1)e^{-3x}$$

$$\lim_{x \rightarrow \infty} -e^{-3x} = - \lim_{x \rightarrow \infty} e^{-3x} = - \lim_{t \rightarrow -\infty} e^t = 0$$

Let $t = -3x$ as $x \rightarrow \infty, t \rightarrow -\infty$

and

$$\lim_{x \rightarrow \infty} e^{-3x} = \lim_{t \rightarrow -\infty} e^t = 0$$

So, by Squeeze Theorem: $\lim_{x \rightarrow \infty} (e^{-3x} \cdot \sin x) = 0$

Where did e come from?

1. Jacob Bernoulli was trying to calculate (in 1618) $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$
2. $e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$
3. $\int_1^e \frac{1}{t} dt = 1$

It was a naturally occurring number (much like π) that just shows up in countless equations that describe the physics of our world (i.e. engineering).

For more information, please click on this link to be taken to the Wikipedia entry for e .

<http://en.wikipedia.org/wiki/E_%28mathematical_constant%29>