Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[10 points]** Give a contrapositive proof for the following. Suppose $z \in \mathbb{R}$. If $z \neq 1$ and $z \neq 4$, then $z^2 + 4 \neq 5z$.

We show that if $z^2 + 1 = 5z$, then $z = 1$ or $z = 4$. Indeed, since $z^2 + 4 = 5z$, we have that $z^2 - 5z + 4 = 0$ and so $(z-4)(z-1) = 0$. It follows that $z = 1$ or $z = 4$.

$\blacksquare$

2. **[10 points]** Let $a, b, a', b' \in \mathbb{Z}$ and let $m \in \mathbb{N}$. Show that if $a \equiv a' \pmod{m}$ and $b \equiv b' \pmod{m}$, then $a + b \equiv a' + b' \pmod{m}$.

Since $a \equiv a' \pmod{m}$, and $b \equiv b' \pmod{m}$, we have $m \mid a - a'$ and $m \mid b - b'$. By definition, this means $a - a' = k_1 m$ and $b - b' = k_2 m$ for some $k_1, k_2 \in \mathbb{Z}$. Adding these equations gives $(a - a') + (b - b') = k_1 m + k_2 m$, which becomes $(a + b) - (a' + b') = (k_1 + k_2) m$ after rearranging terms. Therefore $m \mid (a+b) - (a'+b')$ and it follows that $a + b \equiv a' + b' \pmod{m}$.

3. **[10 points]** Let $x \in \mathbb{R}$. Give a proof by contradiction that x^2 is rational or $(\sqrt{2}) \cdot x$ is irrational.

Suppose for a contradiction that x^2 is irrational and $\sqrt{2}x$ is rational. Since $\sqrt{2}\,x$ is rational, we have that $\sqrt{2}\,x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Squaring both sides gives $(\sqrt{2}\,x)^2 = \frac{a^2}{b^2}$, or $2x^2 = \frac{a^2}{b^2}$. It follows that $x^2 = \frac{a^2}{2b^2}$, and since $a^2, 2b^2 \in \mathbb{Z}$, this implies that x^2 is rational, contradicting our hypothesis that x^2 is irrational. ∎

4. **[2 parts, 10 points each]** Powers of three.

   (a) Let $a, c \in \mathbb{Z}$. Prove that if $3^a < 3^c$, then $\frac{3^a}{3^c} \leq \frac{1}{3}$. (You may use the fact that $f(x) = 3^x$ is an increasing function.)

Scratch: work backward. WANT:

$\frac{3^a}{3^c} \leq \frac{1}{3}$

$\uparrow$

$3^{a+1} \leq 3^c$

$\uparrow$

$a+1 \leq c$

Pf. Since $3^a < 3^c$ and the function $f(x) = 3^x$ is increasing, we have that $a < c$. Since $a, c \in \mathbb{Z}$, $a < c$ implies $a+1 \leq c$. Using again that $f(x) = 3^x$ is increasing, we have that $3^{a+1} \leq 3^c$. This implies $\frac{3^{a+1}}{3^c} \leq 1$ and dividing by 3 gives $\frac{3^a}{3^c} \leq \frac{1}{3}$. ∎

   (b) Use part (a) to show that for all $a, b, c \in \mathbb{Z}$, we have $3^a + 3^b \neq 3^c$.

Suppose for a contradiction that there exist $a, b, c \in \mathbb{Z}$ such that $3^a + 3^b = 3^c$. Since $3^b > 0$, we have that $3^a < 3^a + 3^b = 3^c$, and so $3^a < 3^c$. Similarly, we have $3^b < 3^a + 3^b = 3^c$. It follows from part (a) that $\frac{3^a}{3^c} \leq \frac{1}{3}$ and $\frac{3^b}{3^c} \leq \frac{1}{3}$. Dividing both sides of $3^a + 3^b = 3^c$ by 3^c gives $1 = \frac{3^a}{3^c} + \frac{3^b}{3^c} \leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$, and so we have the contradiction $1 \leq \frac{2}{3}$. ∎

5. **[5 points]** What is the coefficient of x^5y^6 in the expansion of $(x+y)^{11}$? Give a simplified, numerical answer.

By the binomial theorem, this is $\binom{11}{5}$. We compute:

$$\binom{11}{5} = \frac{(11)!}{5!\,(11-5)!} = \frac{(11)!}{(5!)(6!)} = \frac{11\cdot \cancel{10}\cdot \cancel{9}^{3}\cdot \cancel{8}^{2}\cdot 7\cdot \cancel{6!}}{(\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1)\cdot(\cancel{6!})} = 11\cdot 3\cdot 2\cdot 7 = 11\cdot 42 = (10+1)(42) = 420+42 = \boxed{462}.$$

6. **[2 parts, 10 points each]** Algebraic and Combinatorial Proofs. Let $k, n \in \mathbb{Z}$ with $0 \le k \le n$.

(a) Give an algebraic proof that $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$.

We compute $\binom{k}{2} + k(n-k) + \binom{n-k}{2} = \frac{k(k-1)}{2} + k(n-k) + \frac{(n-k)(n-k-1)}{2} = \frac{1}{2}\left[k(k-1) + 2k(n-k) + (n-k)(n-k-1)\right]$

$= \frac{1}{2}\left[k(k-1) + k(n-k) + k(n-k) + (n-k)(n-k-1)\right] = \frac{1}{2}\left[k\big((k-1)+(n-k)\big) + (n-k)\big(k+(n-k-1)\big)\right]$

$= \frac{1}{2}\left[k(n-1) + (n-k)(n-1)\right] = \frac{1}{2}\left[(n-1)(k+(n-k))\right] = \frac{1}{2}\left[(n-1)n\right] = \binom{n}{2}.$

∎

(b) Give a combinatorial proof of the same identity. (Hints: let $U = \{1, \ldots, n\}$. Color k of the integers in U red and the other $n-k$ integers blue. Partition the 2-subsets of U into three groups.)

As in the hint, we color k elements of U red and the remaining $n-k$ elements red. Let $A = \{X \subseteq U : |X| = 2\}$. Let B be the set of all $X \in A$ such that both elements in X are red. Since there are k red elements, $|B| = \binom{k}{2}$. Let D be the set of all $X \in A$ such that both elements in X are blue, and note that $|D| = \binom{n-k}{2}$ since U has $n-k$ blue elements. Let C be the set of all $X \in A$ such that X consists of one red element and one blue element. Since there are k ways to choose the red element and $n-k$ ways to choose the blue element, we have $|C| = k(n-k)$. Since A is the disjoint union of B, C, and D, it follows that

$$\binom{n}{2} = |A| = |B| + |C| + |D| = \binom{k}{2} + k(n-k) + \binom{n-k}{2}.$$

∎

7. **[15 points]** Let $a, b \in \mathbb{Z}$. Show that $b \mid a$ and $b \mid a+1$ if and only if $b = -1$ or $b = 1$.

($\Rightarrow$) Suppose $b|a$ and $b|a+1$. By definition, $a = k_1 b$ and $a+1 = k_2 b$ for some $k_1, k_2 \in \mathbb{Z}$. Subtracting the former from the latter gives

$$(a+1) - a = k_2 b - k_1 b$$

and so $1 = (k_2 - k_1) b$. Since $b | 1$, it follows that $b = -1$ or $b = 1$.

($\Leftarrow$) Let $a, b \in \mathbb{Z}$. Note that $1 | a$ and $-1 | a$ since $a = (1)(a)$ and $a = (-1)(-a)$. It follows that if $b = 1$ or $b = -1$, then $b|a$. ∎

8. **[10 points]** Suppose $a, b, c, d \in \mathbb{R}$. Prove that if $a \neq c$ or $b \neq d$, then there is at most one $x \in \mathbb{R}$ such that $ax + b = cx + d$.

Let $L = \{x \in \mathbb{R} : ax + b = cx + d\}$. We prove the contrapositive: If $|L| \geq 2$, then $a = c$ and $b = d$. Suppose that x_1 and x_2 are distinct elements of L. We have that $ax_1 + b = cx_1 + d$ and $ax_2 + b = cx_2 + d$. Subtracting these gives $(ax_1 + b) - (ax_2 + b) = (cx_1 + d) - (cx_2 + d)$, or $a(x_1 - x_2) = c(x_1 - x_2)$. Since $x_1 \neq x_2$, we have $x_1 - x_2 \neq 0$ and so we may divide both sides by $x_1 - x_2$ to obtain $a = c$. Since $a = c$, we have $ax_1 = cx_1$ and so $ax_1 + b = cx_1 + d$ implies $b = d$. ∎