Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[6 parts, 4 points each]** First, express the following statements using the standard logical operators $\vee, \wedge, \sim, \Rightarrow, \Leftrightarrow$ and given open sentences. Second, state whether the statement is true or false (write the entire word); no justification necessary.

$P_1 : 2+3=8$ $\quad$ P_2 : red is a color $\quad$ $Q(X) : X$ is an infinte set

$R(x) : x$ is prime $\quad$ $S(x) : x$ is odd

(a) Red is a color and $2+3=8$.

$\boxed{P_2 \wedge P_1 \quad \text{FALSE}}$

(b) $2+3 \neq 8$.

$\boxed{\sim P_1 \quad \text{TRUE}}$

(c) If 3 is not odd, then red is not a color.

$\boxed{\sim S(3) \Rightarrow \sim P_2}$

FALSE $\Rightarrow$ FALSE, so $\boxed{\text{TRUE}}$

(d) The integer 5 is odd if and only if $\mathbb{Z}$ is an infinite set.

$\boxed{S(5) \Leftrightarrow Q(\mathbb{Z})}$

TRUE $\Leftrightarrow$ TRUE so $\boxed{\text{TRUE}}$

(e) For 23 to be prime, it is sufficient that 8 is odd.

$\boxed{S(8) \Rightarrow R(23)}$

FALSE $\Rightarrow$ TRUE so $\boxed{\text{TRUE}}$

(f) Either 21 is prime or 9 is odd, but not both.

$\boxed{\sim(R(21) \Leftrightarrow S(9))}$

$\sim($ FALSE $\Leftrightarrow$ TRUE$)$

$\sim$ (FALSE)

$\boxed{\text{TRUE}}$

Also ok:

$(R(21) \vee S(9)) \wedge (\sim R(21) \vee \sim S(9))$

$(R(21) \wedge \sim S(9)) \vee (\sim R(21) \wedge S(9))$

2. **[2 points]** What 1900-era discovery prompted an overhaul of formal mathematics, and why?

Russell's paradox involves the construction of a set $R = \{A : A \text{ is a set and } A \notin A\}$ which leads to a contradiction since both $R \in R$ and $R \notin R$ are impossible. If a system of math has a contradiction, then everything can be proved so the system is not useful.

3. Truth table

   (a) **[6 points]** Give a truth table for $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$.

P	Q	$(P \Rightarrow Q)$	$(Q \Rightarrow P)$	$(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

   (b) **[4 points]** Find a simple formula which is equivalent to $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$.

The formula is equivalent to $\boxed{Q \Rightarrow P}$ since these columns have the same truth values in each row.

4. Let ϕ_1 be the formula $(P \Rightarrow Q) \Rightarrow R$ and let ϕ_2 be the formula $P \Rightarrow (Q \Rightarrow R)$.

   (a) **[6 points]** Find a setting of truth values for P, Q, and R that makes ϕ_1 false and ϕ_2 true.

ϕ_1: $(P \Rightarrow Q) \Rightarrow R$. To be false, we need R false and $P \Rightarrow Q$ true, so either P false or Q true.

ϕ_2: $P \Rightarrow (Q \Rightarrow R)$. To be true, we need P false or $Q \Rightarrow R$ true, so we need P false or Q false, or R true.

So: $\sim R \wedge (\sim P \vee \sim Q) \wedge (\sim P \vee Q)$. Need: R and P false.

Settings:

P	Q	R
F	T	F
F	F	F

   (b) **[4 points]** Based on part (a), what can we conclude about ϕ_1 and ϕ_2?

The formulas ϕ_1 and ϕ_2 are $\boxed{\text{not equivalent}}$.

5. **[5 points]** Consider the following definition: "An integer n is *large* if it takes more than 20 seconds to write down n in decimal." What is problematic about this definition? How can those problems be addressed?

A definition must be precise. Different people may take different times to write down n. Even if the definition was more specific about who or what should write n, the amount of time to write n may not be the same each time. To fix the definition, we should give a threshold like "n is large if $n \geq 10^{30}$".

6. **[4 parts, 4 points each]** First, translate the following statements in formal logic to English as naturally as possible. Second, state whether the statement true or false (write the entire word); give brief justifications where appropriate for partial credit.

   (a) $\exists n \in \mathbb{Z}, (\exists s \in \mathbb{Z}, n = 2s) \wedge (\exists t \in \mathbb{Z}, n = 2t + 1)$

   There is an integer n which is both even and odd. This is false, since $2s = 2t+1$ implies $s - t = \frac{1}{2}$, which is not possible since $s, t \in \mathbb{Z}$.

   (b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y < 0$

   Every real number can be added to some real number to produce a negative number. This is true, since if $x \in \mathbb{R}$, adding $-(1+x)$ gives a negative.

   (c) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y < 0$

   There is a real number with the property that adding it to any real number gives a negative. This is false. For all $x \in \mathbb{R}$, adding $-x$ fails to give a negative number.

   (d) $\forall m \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq m \wedge (\forall k \in \mathbb{N}, 1 < k < n \Rightarrow \frac{n}{k} \notin \mathbb{N})$

   There are infinitely many prime numbers. This is true; we will see a proof later in class.

7. **[3 parts, 3 points each]** Negate each sentence below in English as naturally as possible.

   (a) Some integer is both a perfect square and a perfect cube.

   Every integer fails to be a perfect square or fails to be a perfect cube.

   (b) Every set of real numbers has a positive real number as a member.

   There exists a set of real numbers such that each member is at most 0.

   (c) For each nonempty set A of real numbers, if $a \leq 100$ for each $a \in A$, then there exists $M \in A$ such that $b \leq M$ for each $b \in A$.

   There exists a nonempty set A of real numbers such that $a \leq 100$ for each $a \in A$ and for each $M \in A$, some $b \in A$ is larger than M.

8. **[8 points]** Prove that if n is an odd integer, then $n^2 - 1$ is a multiple of 4.

Pf. Since n is odd, we have that $n = 2k+1$ for some $k \in \mathbb{Z}$. We compute

$$
\begin{aligned}
n^2 - 1 &= (2k+1)^2 - 1 \\
&= 4k^2 + 4k + 1 - 1 \\
&= 4k(k+1).
\end{aligned}
$$

Since $k(k+1) \in \mathbb{Z}$, it follows that $4 \mid n^2 - 1$, and so $n^2 - 1$ is a multiple of 4. ∎

9. **[2 parts, 8 points each]** A two-step proof. In both parts, let a, b, and d be integers.

(a) Prove that if $d \mid b$ and $d \mid a + b$, then $d \mid a$.

Pf. Suppose $d \mid b$ and $d \mid a+b$. This means that $b = k_1 d$ and $a+b = k_2 d$ for some $k_1, k_2 \in \mathbb{Z}$. Subtracting the first equation from the second gives

$$(a+b) - b = k_2 d - k_1 d$$

which simplifies to $a = (k_2 - k_1)d$. Since $k_2 - k_1 \in \mathbb{Z}$, we have $d \mid a$. ∎

(b) Use part (a) to show that if $d \mid an + b$ for each $n \in \mathbb{Z}$, then $d \mid a$ and $d \mid b$.

Suppose that $d \mid an+b$ for each $n \in \mathbb{Z}$. When $n = 0$, we have that $d \mid b$ and when $n = 1$, we have that $d \mid a+b$. It follows from part (a) that d also divides a. Hence $d \mid a$ and $d \mid b$. ∎