Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. Prove the following using a direct proof.

   (a) **[4 points]** Let $a, b \in \mathbb{Z}$. We have that $a + b$ is even if and only if $a - b$ is even.

<u>Pf</u> First, suppose $a+b$ is even. Therefore $a+b = 2k$ for some $k \in \mathbb{Z}$. Adding $-2b$ to both sides gives $a-b = 2k-2b$, and so $a-b = 2(k-b)$. Since $k-b$ is an integer, it follows that $2 \mid a-b$ and so $a-b$ is even.

Conversely, suppose $a-b$ is even. Therefore $a-b = 2k$ for some $k \in \mathbb{Z}$. Now we add $2b$ to both sides to get $a+b = 2k+2b = 2(k+b)$. Since $k+b \in \mathbb{Z}$, it follows that $a+b$ is even. ∎

   (b) **[3 points]** Let $x \in \mathbb{R}$. If $x \neq 1$, then $x < \frac{x^2+1}{2}$.

<u>IDEA</u>: work backwards. Want:

$$x < \frac{x^2+1}{2}$$
$$2x < x^2+1$$
$$0 < x^2-2x+1$$
$$0 < (x-1)^2.$$

<u>Pf</u>. Since $x \neq 1$, it follows that $x - 1 \neq 0$ and so $(x-1)^2$ is strictly positive. Since $0 < (x-1)^2$ and $(x-1)^2 = x^2-2x+1$, it follows that $0 < x^2-2x+1$. Adding $2x$ to both sides gives $2x < x^2+1$ and dividing by 2 gives $x < \frac{x^2+1}{2}$. ∎

(c) **[3 points]** Let $a \in \mathbb{Z}$. If $5 \mid 8a$, then $5 \mid a$. (Hint: the equation $a = 16a - 15a$ may be useful.)

Pf. Since $5 \mid 8a$, we have $8a = 5k$ for some $k \in \mathbb{Z}$. Starting with $a = 16a - 15a$, we compute:

$$\begin{aligned} a &= 16a - 15a \\ &= 2(8a) - 15a \\ &= 2(5k) - 15a \\ &= 5(2k - 3a). \end{aligned}$$

Since $a = 5(2k-3a)$ and $2k - 3a \in \mathbb{Z}$, it follows that $5 \mid a$. $\blacksquare$