Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[3 parts, 1 point each]** Translate the following sentences to symbolic logic as directly and simply as possible. Is the statement true or false? Write the entire word.

   (a) Adding 1 to each positive real number results in a positive real number.

   $\boxed{\forall x \in \mathbb{R}, \quad x > 0 \Rightarrow x+1 > 0}$ $\boxed{\text{TRUE}}$

   (b) Every real number can be multiplied by some real number to produce a rational number.

   $\boxed{\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \quad xy \in \mathbb{Q}}$ $\boxed{\text{TRUE}}$ For all $x \in \mathbb{R}$, we may take $y=0$ and $x \cdot 0 = 0 \in \mathbb{Q}$.

   (c) The only set which is a subset of every set is the empty set.

   $\forall$ sets A, $(\forall$ sets B, $A \subseteq B) \Rightarrow A = \emptyset$. $\boxed{\text{TRUE}}$. To be a subset of every set requires being a subset of $\emptyset$, and only $\emptyset$ is a subset of $\emptyset$.

2. **[3 parts, 1 point each]** Translate the following statements/open sentences in symbolic logic to English sentences as simply as possible. Is the statement true or false? Write the entire word.

   (a) $\sim (\exists x \in \mathbb{Q}, x^2 = 2)$

   $\boxed{\text{The number } \sqrt{2} \text{ is irrational.}}$ $\boxed{\text{TRUE}}$. (We've seen this example in class.)

   (b) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \exists k \in \mathbb{Z}, x + y = 2k$

   $\boxed{\text{There is an integer which, when added to any integer, produces an even integer.}}$

   $\boxed{\text{FALSE}}$. For all $x \in \mathbb{Z}$, either $x+0$ or $x+1$ is odd.

   (c) $\forall A \subseteq \mathbb{N}, \forall B \subseteq \mathbb{N}, (\exists a \in \mathbb{N}, |A| \leq a) \wedge (\exists b \in \mathbb{N}, |B| \leq b) \implies (\exists c \in \mathbb{N}, |A \cup B| \leq c)$

   $\boxed{\text{The union of two finite sets is finite.}}$ $\boxed{\text{True}}$

3. **[2 parts, 2 points each]** Negate the following sentences as simply and naturally as possible. (You may translate to and from symbolic logic if helpful, but this is not required.) Is the **original statement** true or false? Explain.

   (a) There are at least 7 prime numbers or 5 is less than 0.

Negation: There are fewer than 7 prime numbers and 5 is at least 0.

The original statement is $\boxed{\text{true}}$ because there are infinitely many primes (so at least 7.)

   (b) Every infinite subset of integers contains an infinite subset of even integers.

Orig: $\forall A \subseteq \mathbb{Z}$, A infinite $\Rightarrow$ $\exists B \subseteq A$, B infinite $\wedge$ $B \subseteq \{x \in \mathbb{Z} : x \text{ is even}\}$

Neg: $\exists A \subseteq \mathbb{Z}$, $(A \text{ infinite}) \wedge \sim(\exists B \subseteq A,\ B \text{ infinite} \wedge B \subseteq \{x \in \mathbb{Z} : x \text{ is even}\})$

$\exists A \subseteq \mathbb{Z}$, $(A \text{ infinite}) \wedge$ $\forall B \subseteq A$ B finite or $B \not\subseteq \{x \in \mathbb{Z} : x \text{ is even}\}$

$\exists A \subseteq \mathbb{Z}$, $(A \text{ infinite}) \wedge$ $\forall B \subseteq A$ B finite or B contains an odd integer

$\exists A \subseteq \mathbb{Z}$, $(A \text{ infinite}) \wedge$ $\forall B \subseteq A$ B infinite $\Rightarrow$ B contains an odd integer.

There is an infinite set of integers such that every infinite subset contains an odd integer.

The original statement is $\boxed{\text{false}}$. Indeed, the set $\{x \in \mathbb{Z} : x \text{ is odd}\}$ shows that the negation of the statement is true.