**Math 303** **Quiz 12** **Fri. Apr. 19, 2024**

Name: Solutions

**Directions:** Solve the following problems. Give supporting work/justification where appropriate.

1. [**4 points**] Let $A$ and $B$ be sets. Prove that $A \subseteq B$ if and only if $A \cup B = B$.

($\Rightarrow$) Suppose $A \subseteq B$. We show that $A \cup B = B$. First, we argue $A \cup B \subseteq B$. Let $x \in A \cup B$. This means $x \in A$ or $x \in B$. If $x \in A$, then $A \subseteq B$ implies $x \in B$ also. In the other case, $x \in B$ directly. Since each element in $A \cup B$ is in $B$, we have $A \cup B \subseteq B$. Next, we argue $A \cup B \supseteq B$. Let $x \in B$. Since $x \in A$ or $x \in B$, we have $x \in A \cup B$. It follows that $B \subseteq A \cup B$. Since $A \cup B \subseteq B$ and $A \cup B \supseteq B$, we have that $A \cup B = B$.

($\Leftarrow$) Suppose that $A \cup B = B$. We show that $A \subseteq B$. Let $x \in A$. We have $x \in A$ or $x \in B$, and so $x \in A \cup B$. Since $A \cup B = B$ and $x \in A \cup B$, it follows that $x \in B$. Since each element in $A$ is also in $B$, we have $A \subseteq B$. $\blacksquare$

2. **[3 points]** Prove or disprove: Let $A$ and $B$ be sets. We have $(A \times B) \cup (B \times A) = (A \cup B) \times (A \cup B)$.

Disproof. Let $A = \{1\}$ and $B = \emptyset$. Note that $A \times B = B \times A = \emptyset$, and so $(A \times B) \cup (B \times A) = \emptyset \cup \emptyset = \emptyset$. On the other hand, $A \cup B = \{1\}$ and so $(A \cup B) \times (A \cup B) = \{1\} \times \{1\} = \{(1,1)\}$. Since $(A \times B) \cup (B \times A) \neq (A \cup B) \times (A \cup B)$, this $A$ and $B$ form a counter-example.

3. **[3 points]** Prove or disprove: Let $A$ and $B$ be sets. We have $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$.

Proof. ($\subseteq$) Suppose $(x,y) \in (A \times B) \cap (B \times A)$. We have $(x,y) \in A \times B$ and $(x,y) \in B \times A$. Since $(x,y) \in A \times B$, this means $x \in A$ and $y \in B$. Since $(x,y) \in B \times A$, this means $x \in B$ and $y \in A$. Since $x \in A$ and $x \in B$, we have $x \in A \cap B$. Similarly, since $y \in B$ and $y \in A$, we have $y \in A \cap B$. Therefore $(x,y) \in (A \cap B) \times (A \cap B)$. It follows that $(A \times B) \cap (B \times A) \subseteq (A \cap B) \times (A \cap B)$.

($\supseteq$). Suppose $(x,y) \in (A \cap B) \times (A \cap B)$. We have $x \in A \cap B$ and $y \in A \cap B$. It follows that $x \in A$, $x \in B$, $y \in A$, and $y \in B$. Hence $(x,y) \in A \times B$ and $(x,y) \in B \times A$. Therefore $(x,y) \in (A \times B) \cap (B \times A)$. Since each element in $(A \cap B) \times (A \cap B)$ is also in $(A \times B) \cap (B \times A)$, it follows that $(A \times B) \cap (B \times A) \supseteq (A \cap B) \times (A \cap B)$. ∎