**Directions:** You may work to solve these problems in groups, but all written work must be your own. Unless the problem indicates otherwise, all problems require some justification; a correct answer without supporting reasoning is not sufficient. Submissions must be stapled. See "Guidelines and advice" on the course webpage for more information.

1. Prove the following using induction or a "no smallest counterexample" argument.
   (a) For each $n \in \mathbb{N}$, we have $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
   (b) For each $n \in \mathbb{N}$, we have $\displaystyle\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$.
2. Use induction to prove the following.
   (a) For each $n \in \mathbb{N}$, we have $\sum_{i=1}^{n}(8i-5) = 4n^2 - n$.
   (b) For each non-negative integer $n$, we have $9 \mid 4^{3n} + 8$.
   (c) If $n \in \mathbb{N}$, then $2^n + 3^n \leq 5^n$.
   (d) If $n \in \mathbb{N}$, then $(\sum_{k=1}^{n} k)^2 = \sum_{j=1}^{n} j^3$.
   (e) If $n \in \mathbb{N}$, then $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$. (Hint: you may need one to use one of our identities involving binomial coefficients.)
3. Using any method, prove the following.
   (a) If $x_1, \ldots, x_n$ are non-negative real numbers, then
   $$(1+x_1)(1+x_2)\cdots(1+x_n) \geq 1 + x_1 + x_2 + \cdots + x_n.$$
   (b) If $n \in \mathbb{Z}$, then $\gcd(3n+5, 5n+8) = 1$.
   (c) If $n$ is a non-negative integer, then $(1+\sqrt{2})^n + (1-\sqrt{2})^n$ is an even integer.
4. Recall the Fibonacci numbers, defined by $F_1 = 1$, $F_2 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$. Prove that each integer $n$ can be represented as the sum of Fibonacci numbers, no two of which are consecutive. (For example, $2 = F_3$, $10 = F_6 + F_3 = 8 + 2$, and $20 = F_7 + F_5 + F_3 = 13 + 5 + 2$. However, even though $20 = F_7 + F_5 + F_2 + F_1$, this is not a desired representation for 20 since it uses the consecutive Fibonacci numbers $F_1$ and $F_2$.)