Directions: You may work to solve these problems in groups, but all written work must be your own. Unless the problem indicates otherwise, all problems require some justification; a correct answer without supporting reasoning is not sufficient. Submissions must be stapled. See "Guidelines and advice" on the course webpage for more information.

1. Prove the following by contradiction.
   (a) If $b \in \mathbb{Z}$ and $b \nmid k$ for every $k \in \mathbb{N}$, then $b = 0$.
   (b) Suppose $a, b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

2. An identity. Let $n \geq 1$ and let $[n] = \{1, \ldots, n\}$.
   (a) Using the binomial theorem, show that $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$. (Hint: substitute appropriate values for x and y in the equation from the binomial theorem.)
   (b) By pairing the even subsets of $[n]$ with distinct odd subsets of $[n]$ (that is, each subset should have its own distinct partner subset of opposite parity), give a combinatorial proof that $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

3. Prove the following (using any method).
   (a) There is a prime number between 90 and 100.
   (b) If $a \in \mathbb{Z}$, then $a^3 \equiv a \pmod{3}$.
   (c) There exists a positive real number x for which $x^2 < \sqrt{x}$.
   (d) Suppose $a, b \in \mathbb{Z}$. If $a + b$ is odd, then $a^2 + b^2$ is odd.
   (e) There exist unique non-negative integers x and y such that $145408 = x \cdot 2^y$ and x is odd.

4. Let $a, b, c, d \in \mathbb{Z}$ and suppose that $bc - ad = 1$. Prove that $\gcd(an + b, cn + d) = 1$ for each $n \in \mathbb{Z}$. Hint: show that $bc - ad$ is an integer combination of $an + b$ and $cn + d$. That is, find integers x and y such that $x(an + b) + y(cn + d) = bc - ad = 1$.

5. Let n be an integer such that $n \geq 3$, and suppose that n lights are arranged in a circle. Initially, all lights are off. Each light is attached to a switch, but flipping a switch toggles the on/off status of its light and the two neighboring lights. In terms of n, what is the minimum number of switch flips needed to turn all lights on? Prove that your answer is correct.