Directions: You may work to solve these problems in groups, but all written work must be your own. Unless the problem indicates otherwise, all problems require some justification; a correct answer without supporting reasoning is not sufficient. Submissions must be stapled. See "Guidelines and advice" on the course webpage for more information.

1. Prove the following using the method of proof by contradiction.
   (a) Show that $2^{\frac{1}{3}}$ is irrational.
   (b) Suppose that $a, b, c \in \mathbb{Z}$. Show that if $a^2 + b^2 = c^2$, then a or b is even.
   (c) Prove that there are no integers a and b such that $21a + 30b = 1$.

2. Irrational powers of three.
   (a) Let a be an integer. Prove that if $3 \mid a^2$, then $3 \mid a$.
   (b) Prove that if k is an odd positive integer, then $\sqrt{3^k}$ is irrational. Hint: suppose for a contradiction that the implication is false for some values of k, and let k be the least odd positive integer for which the implication fails.

3. Using only logic and trigonometry (not calculus), show that $\sin(x) + \sqrt{3}\cos(x) \leq 2$ for each real number x. (Hint: recall that $\tan(\pi/3) = \sqrt{3}$.)

4. Critique the following argument. (Be careful!)

   Theorem 1. *If $p_1, \ldots, p_k$ is a list of the first k primes, then $p_1 p_2 \cdots p_k + 1$ is also a prime.*

   Proof: Let $n = p_1 p_2 \cdots p_k + 1$, and note that $1 = n - p_1 p_2 \cdots p_k$.
   Suppose for a contradiction that some prime p_i less than n divides n. If this were true, then p_i divides both terms on the right hand side of $1 = n - p_1 p_2 \cdots p_k$ and therefore p_i must also divide the left hand side of this equation. Since no prime divides 1, we have a contradiction.
   The contradiction implies that no prime less than n divides n, and therefore n is prime. □

5. Counting Subsets and The Binomial Theorem.
   (a) Suppose that A is a set and $|A| = 84$. How many subsets of A have 0 elements? How many have 10 elements? How many have 74 elements?
   (b) Suppose that A is a set and there are 330 subsets of A of size 7. What is $|A|$?
   (c) Use the binomial theorem to find the coefficient of $x^4 y^8$ in $(3x - 2y)^{12}$.

6. Prove the following (using any method).
   (a) Suppose that $x \in \mathbb{Z}$. Prove that x is even if and only if $3x + 5$ is odd.
   (b) An integer a is odd if and only if a^3 is odd.
   (c) Suppose $x, y \in \mathbb{R}$. Then $(x + y)^2 = x^2 + y^2$ if and only if $x = 0$ or $y = 0$.

7. Prove that for each $d \in \mathbb{N}$, there exist infinitely many primes p such that $p + d$ is not prime. (Hint: try proof by contradiction. What does it mean for this claim to be false? In other words, what is the logical negation of this claim? If you have trouble formulating the negation, then it may be helpful to translate the claim into a symbolic logic formula φ, negate φ and simplify, and finally translate back into English.)