Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[2.5 points]** Let a, b, and c be integers. Prove that if $a \mid b$ and $b \mid c$, then $a \mid c$.

Since $a|b$, we have $b = ka$ for some $k \in \mathbb{Z}$. Since $b|c$, we have $c = lb$ for some $l \in \mathbb{Z}$. It follows that

$$c = lb = l(ka) = (lk)a.$$

Since lk and a are integers, we conclude that $a|c$. ∎

2. **[2.5 points]** Let $a = 9169$, $b = 1007$, and let $d = \gcd(a, b)$. Use the extended Euclidean algorithm to compute d and integers u and v such that $d = ua + vb$.

$$9169 = (9)(1007) + 106$$

$$1007 = (9)(106) + 53$$

$$106 = (2)(53) + 0$$

$$\begin{array}{r} 1007 \\ 954 \\ \hline 53 \end{array}$$

$$d = \gcd(a,b) = \gcd(53, 0) = 53.$$

$$\begin{aligned} 53 &= (1)(1007) + (-9)(106) \\ &= (1)(1007) + (-9)[9169 - (9)(1007)] \\ &= \underbrace{(82)}_{v}(1007) + \underbrace{(-9)}_{u}(9169). \end{aligned}$$

So $\boxed{(d, u, v) = (53, -9, 82)}$ and $\boxed{53 = (-9)(9169) + (82)(1007).}$

3. **[2 parts, 2.5 points each]** Consider the following 1-person game, played on the number line. Initially, the player begins at 0. At each step, the player can move 525 units or 462 units in either direction.

   (a) What is the smallest positive integer on which the player can land, and why?

Let $d = \gcd(525, 462)$:

$$525 = (1)(462) + 63$$
$$462 = (7)(63) + 21$$
$$63 = (3)(21) + 0$$
$$\gcd(525, 462) = 21$$

$$21 = 462 + (-7)(63)$$
$$= 462 + (-7)[525 + (-1)(462)]$$
$$= (8)(462) + (-7)(525).$$

$$\begin{array}{r} 525 \\ 462 \\ \hline 63 \end{array} \qquad \begin{array}{r} 63 \\ 6 \\ \hline 378 \end{array} \qquad \begin{array}{r} \cancel{4}62 \\ 378 \\ \hline 84 \end{array}$$

The smallest pos. integer is $\boxed{21}$. Since 21 is a common multiple of 525 and 462, every integer that can be reached is a multiple of 21, so no number in $\{1, \ldots, 20\}$ is reachable. Also, by taking 8 steps of 462 units in the positive direction and 7 steps of 525 units in the negative direction, we reach 21.

   (b) What is the smallest positive *even* integer on which the player can land, and why?

This is $\boxed{42}$, which equals 2 times 21. As in part (a), every reachable number is a multiple of 21, and the smallest positive even multiple of 21 is 42. We can reach 42 via doubling the equation that gets us to 21:

$$21 = (-7)(525) + (8)(462)$$
$$2 \cdot 21 = (2 \cdot (-7))(525) + (2 \cdot 8)(462)$$
$$\boxed{42 = (-14)(525) + (16)(462)}$$