Name: _____

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[2 parts, 10 points each]** Give a contrapositive proof of the following.

   (a) Suppose $a, b \in \mathbb{Z}$. If $ab + b$ is even, then a is odd or b is even.

   (b) Suppose $x \in \mathbb{R}$. If $x^2 + 5x < 0$, then $x < 0$.

2. **[10 points]** Prove the following. Let $a \in \mathbb{Z}$. If $a \equiv 3 \pmod 7$, then $a^2 \equiv 2 \pmod 7$.

3. **[10 points]** Prove that for each $x \in \mathbb{R}$, either $(x+\sqrt{2})$ is irrational or $(-x+\sqrt{2})$ is irrational.

4. **[10 points]** Let x and y be positive real numbers. Prove that if $x \neq y$, then $\frac{x}{y} + \frac{y}{x} > 2$.

5. **[10 points]** Let $a, b, c \in \mathbb{Z}$. Use the corollary below to prove that if $a \mid c$ and $b \mid c$ where $\gcd(a, b) = 1$, then $ab \mid c$.

   Corollary 1. *Let $x, y, z \in \mathbb{Z}$. If $x \mid yz$ and $\gcd(x, y) = 1$, then $x \mid z$.*

6. **[5 points]** How many subsets of $\{1, \ldots, 14\}$ have size 4? Give a simplified, numerical answer.

7. **[3 parts, 5 points each]** A business class has a total enrollment of 26 students, with 14 men and 12 women. The class will send a team of 6 students to compete in a national contest. In the following, you may leave your answers in terms of binomial coefficients and simple arithmetic operations (no need to simplify).

   (a) How many ways are there to select a team?

   (b) How many ways are there to select a team consisting of all women?

   (c) How many ways are there to select a team with at least one man and at least one woman?

8. **[20 points]** Let n be a positive integer. Prove that there exist unique non-negative integers a and b such that $n = 3^a \cdot b$ and $3 \nmid b$.

(Scratch Paper)