Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[2 parts, 10 points each]** Give a contrapositive proof of the following.

   (a) Suppose $a, b \in \mathbb{Z}$. If $ab + b$ is even, then a is odd or b is even.

We show that if a is even and b is odd, then $ab+b$ is odd. Since a is even, we have $a = 2k$ for some $k \in \mathbb{Z}$. Since b is odd, we have $b = 2l+1$ for some $l \in \mathbb{Z}$. It follows that $ab + b = (2k)(2l+1) + (2l+1) = 4kl + 2k + 2l + 1$, or $ab + b = 2(2kl + k + l) + 1$. Since $2kl + k + l$ is an integer, it follows that $ab + b$ is odd. ∎

   (b) Suppose $x \in \mathbb{R}$. If $x^2 + 5x < 0$, then $x < 0$.

We prove that if $x \geq 0$, then $x^2 + 5x \geq 0$. Since $x \geq 0$, we may multiply both sides by the non-negative number 5 to obtain $5x \geq 5 \cdot 0$ or $5x \geq 0$. Similarly, we may multiply both sides of $x \geq 0$ by x to obtain $x^2 \geq 0$. Adding both $x^2 \geq 0$ and $5x \geq 0$ gives $x^2 + 5x \geq 0 + 0$, and so $x^2 + 5x \geq 0$. ∎

2. **[10 points]** Prove the following. Let $a \in \mathbb{Z}$. If $a \equiv 3 \pmod 7$, then $a^2 \equiv 2 \pmod 7$.

We give a direct proof. Suppose $a \equiv 3 \pmod 7$. This means $7 \mid a - 3$, and so $a - 3 = 7k$ for some $k \in \mathbb{Z}$. So $a = 7k + 3$ and $a^2 = (7k+3)^2 = 49k^2 + 42k + 9$.

Note that $a^2 - 2 = 49k^2 + 42k + 9 - 2 = 7(7k^2 + 6k + 1)$. Since $7k^2 + 6k + 1 \in \mathbb{Z}$, we have that $7 \mid a^2 - 2$, and so $a^2 \equiv 2 \pmod 7$ by definition. ∎

3. **[10 points]** Prove that for each $x \in \mathbb{R}$, either $(x+\sqrt{2})$ is irrational or $(-x+\sqrt{2})$ is irrational.

Suppose for a contradiction that $x \in \mathbb{R}$ and both $x+\sqrt{2}$ and $-x+\sqrt{2}$ are rational. This means $x+\sqrt{2} = \frac{a}{b}$ and $-x+\sqrt{2} = \frac{c}{d}$ for some $a,b,c,d \in \mathbb{Z}$.

Adding both equations gives $(x+\sqrt{2})+(-x+\sqrt{2}) = \frac{a}{b}+\frac{c}{d}$, and this simplifies to $2\sqrt{2} = \frac{ad+bc}{bd}$. After dividing by 2, we have $\sqrt{2} = \frac{ad+bc}{2bd}$. Since both $ad+bc$ and $2bd$ are integers, this implies $\sqrt{2}$ is rational. But we know from class that $\sqrt{2}$ is irrational, so this is a contradiction.

Scratch work for #4:

$\frac{x}{y}+\frac{y}{x} > 2$

$\frac{x^2}{yx}+\frac{y^2}{yx} > 2$

$x^2+y^2 > 2yx$

$x^2-2yx+y^2 > 0$

$(x-y)^2 > 0$

Now start here in the proof and work backward.

4. **[10 points]** Let x and y be positive real numbers. Prove that if $x \neq y$, then $\frac{x}{y}+\frac{y}{x} > 2$.

Suppose that x and y are positive distinct real numbers.

Since x and y are distinct, we have $(x-y)^2 > 0$.

It follows that $x^2-2xy+y^2 > 0$ and so $x^2+y^2 > 2xy$. Since x and y are both positive, so is xy. Dividing both sides by xy gives

$$\frac{x^2}{xy}+\frac{y^2}{xy} > 2$$

which simplifies to $\frac{x}{y}+\frac{y}{x} > 2$. ∎

5. **[10 points]** Let $a, b, c \in \mathbb{Z}$. Use the corollary below to prove that if $a \mid c$ and $b \mid c$ where $\gcd(a, b) = 1$, then $ab \mid c$.

   Corollary 1. *Let $x, y, z \in \mathbb{Z}$. If $x \mid yz$ and $\gcd(x, y) = 1$, then $x \mid z$.*

Suppose $a|c$ and $b|c$ where $\gcd(a,b)=1$. Since $a|c$ and $b|c$, we have $c=ak$ and $c=bl$ for some $k,l \in \mathbb{Z}$. So $ak=bl$ and since all factors are integers, we have $a|bl$. Since $\gcd(a,b)=1$, the Corollary applies with $x=a$, $y=b$, and $z=l$. It follows that $a|l$, and so $l=ta$ for some $t \in \mathbb{Z}$. So $c=bl=b(ta)=t(ab)$ and it follows that $ab|c$. ▨

6. **[5 points]** How many subsets of $\{1, \ldots, 14\}$ have size 4? Give a simplified, numerical answer.

$$\binom{14}{4} = \frac{14!}{4!\,(14-4)!} = \frac{14!}{4!\,(10)!} = \frac{\overset{7}{\cancel{14}}\cdot 13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10!}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1\cdot\cancel{10!}} = 7\cdot 13\cdot 11 = 91\cdot 11 = 910+91 = \boxed{1001}$$

7. **[3 parts, 5 points each]** A business class has a total enrollment of 26 students, with 14 men and 12 women. The class will send a team of 6 students to compete in a national contest. In the following, you may leave your answers in terms of binomial coefficients and simple arithmetic operations (no need to simplify).

   (a) How many ways are there to select a team?

$$\boxed{\binom{26}{6}}$$

From 26 total students, pick 6.

   (b) How many ways are there to select a team consisting of all women?

$$\boxed{\binom{12}{6}}$$

From 12 women, pick 6.

   (c) How many ways are there to select a team with at least one man and at least one woman?

$$\boxed{\binom{26}{6} - \underbrace{\binom{12}{6}}_{\#\text{ teams all women}} - \underbrace{\binom{14}{6}}_{\#\text{ teams all men}}}$$

From all teams, subtract the teams with all women and the teams with all men

8. **[20 points]** Let n be a positive integer. Prove that there exist unique non-negative integers a and b such that $n = 3^a \cdot b$ and $3 \nmid b$.

Existence: Let a be the maximum integer such that $3^a \mid n$. Note that $a \geq 0$, since $3^0 = 1$ and $1 \mid n$. Since $3^a \mid n$, we have $n = 3^a \cdot b$ for some $b \in \mathbb{Z}$. Since n and 3^a are both positive integers, it follows that b is also positive, and so b is non-negative. We claim that $3 \nmid b$. Indeed, if $3 \mid b$, then $b = 3s$ for some $s \in \mathbb{Z}$ and we would have

$$n = 3^a \cdot b = 3^a \cdot (3s) = 3^{a+1} \cdot s$$

giving $3^{a+1} \mid n$ and contradicting that a is the maximum integer such that $3^a \mid n$. So $3 \nmid b$ as claimed and $n = 3^a \cdot b$ for some non-negative integers a and b such that $3 \nmid b$.

Uniqueness: Suppose $n = 3^{a_1} \cdot b_1$ and $n = 3^{a_2} \cdot b_2$ where a_1, a_2, b_1, b_2 are all non-negative integers, $3 \nmid b_1$, and $3 \nmid b_2$. We may assume, without loss of generality, that $a_1 \geq a_2$. We have $3^{a_1} \cdot b_1 = 3^{a_2} \cdot b_2$, and dividing both sides by 3^{a_2} gives $3^{a_1 - a_2} \cdot b_1 = b_2$. Note that if $a_1 > a_2$, then 3 divides $3^{a_1 - a_2} \cdot b_1$ and so $3 \mid b_2$. But we know $3 \nmid b_2$, and so it follows that $a_1 = a_2$. Hence $3^{a_1 - a_2} \cdot b_1 = b_2$ simplifies to $3^0 \cdot b_1 = b_2$, or $1 \cdot b_1 = b_2$. Since $a_1 = a_2$ and $b_1 = b_2$, there is only one pair of non-negative integers a, b such that $n = 3^a \cdot b$ and $3 \nmid b$. ∎

(Scratch Paper)