Name: ____________________

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[6 parts, 4 points each]** For each of the following, decide whether the given string of symbols forms a sentence. In the case of a sentence, state whether the sentence is true or false. Indicate your answer by writing the entire word "true" or "false".

P : 9 is a square number $\qquad$ $Q(x)$: x is even $\qquad$ $S(A)$: A is a finite set

(a) $P \sim \vee\, Q(4)$

(b) $\sim (P \vee Q(5))$

(c) $S(\mathbb{R}) \implies Q(1)$

(d) $\sim (\sim (\sim P))$

(e) $(Q(2) \cap \mathbb{Z}) \wedge (1 + 2 = 6)$

(f) $\forall A \in \mathcal{P}(\mathbb{N}), [A \neq \varnothing \implies \sim S(A)]$

2. **[2 parts, 4 points each]** Translate the following to symbolic logic as directly and simply as possible. Indicate whether the statement is true or false by writing the entire word.

   (a) There is an integer which is within distance $1/2$ to every real number.

   (b) There are real numbers a and b such that $ax = b$ for every real number x.

3. **[2 parts, 4 points each]** Translate the following statements to English as simply as possible. Indicate whether the statement is true or false by writing the entire word.

   (a) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, [(x = 2y) \vee (x + 1 = 2y)]$

   (b) $\forall S \subseteq \mathbb{R}, [(\forall n \in \mathbb{N}, |S| \geq n)) \implies (\exists x \in S, x \notin \mathbb{Q})]$

4. **[2 parts, 4 points each]** For each statement below, give the logical negation in a simple, natural English statement.

   (a) For all integers a and b, if $a^2 \mid b^2$, then $a \mid b$.

   (b) For each continuous function $f(x)$ such that $f(-1) < 0$ and $f(1) > 0$, there is a real number a such that $-1 < a < 1$ and $f(a) = 0$.

5. Equivalent statements.

   (a) [**8 points**] Give a truth table for φ, where φ is $(P \implies Q) \iff (Q \implies P)$.

   (b) [**4 points**] Use the truth table to give a simple statement that is equivalent to φ.

6. [**2 parts, 8 points each**] Critique the following proofs. Is the proof correct? If so, can it be improved? If not, where does the proof go wrong? Can it be fixed by modifying the proof, modifying the statement of the theorem, or both?

   Theorem 1. *If $x \in \mathbb{R}$ and $x^5 > 1$, then $x > 1$.*

   (a) **Proof:** Since $x^5 > 1$, we have that $x^5 - 1 > 0$. The left hand side factors as $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$, giving $(x-1)(x^4 + x^3 + x^2 + x + 1) > 0$. This implies both $x - 1 > 0$ and $x^4 + x^3 + x^2 + x + 1 > 0$. $\square$

   (b) **Proof:** Note that if $x > 1$, then we may multiply both sides of the inequality by the positive number x to get $x^2 > x$. Since $x^2 > x$ and $x > 1$, this implies $x^2 > 1$. Next, we multiply both sides of $x^2 > 1$ by the positive number x to obtain $x^3 > x$, and with $x > 1$ it follows that $x^3 > 1$. Continuing in this way, we see that $x^5 > 1$. Therefore $x^5 > 1$ and $x > 1$ are equivalent statements. Since we assume the hypothesis $x^5 > 1$, the equivalent statement $x > 1$ follows. $\square$

7. **[12 points]** Prove that if x and y are odd integers, then xy is odd.

8. **[12 points]** Prove that if n is an even integer, then $n^2 + 2n$ is a multiple of 8. (Hint: consider appropriate cases.)

(Scratch Paper)