Math 303 **Quiz 9** **Fri. Mar. 31, 2023**

Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. [**2 points**] How many subsets of $\{1, 2, \ldots, 9\}$ have size 4? Give a numerical answer.

$$\binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{9\cdot \overset{2}{\cancel{8}}\cdot 7\cdot \cancel{6}\cdot \cancel{5!}}{(\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1)\cdot \cancel{5!}} = 9\cdot 2\cdot 7 = 9\cdot 14 = (10-1)\cdot 14 = 140 - 14 = \boxed{126}$$

2. [**4 points**] Prove that $\sqrt{5}$ is irrational. You may use that if $a \in \mathbb{Z}$ and $5 \mid a^2$, then $5 \mid a$.

Pf: Suppose for a contradiction that $\sqrt{5}$ is rational. Let a and b be integers such that $\sqrt{5} = \frac{a}{b}$, with a and b written in lowest common terms, so that a and b have no common divisors except 1 and -1. We have $\sqrt{5}\, b = a$, and so $5b^2 = a^2$. It follows that $5 \mid a^2$, and so $5 \mid a$. Since $5 \mid a$, we have $a = 5k$ for some $k \in \mathbb{Z}$. Substituting for a, we get $5b^2 = a^2 = (5k)^2 = 5^2 k^2$. Dividing both sides by 5 gives $b^2 = 5k^2$. So $5 \mid b^2$ and therefore $5 \mid b$. But now we have $5 \mid a$ and $5 \mid b$, contradicting that $\frac{a}{b}$ is in least common terms. ∎

3. [**4 points**] Use a proof by contradiction to show that if $n \in \mathbb{Z}$, then $4 \nmid n^2 + 2$.

Pf: Suppose for a contradiction that $n \in \mathbb{Z}$ but $4 \mid n^2 + 2$. This means $n^2 + 2 = 4k$ for some $k \in \mathbb{Z}$. So $n^2 = 4k - 2 = 2(2k-1)$. Since $2 \mid n^2$ and n^2 is even, it follows that n is also even (since if n were odd, then n^2 is the result of multiplying an odd number times an odd number, which would make n^2 odd). Since n is even, we have $n = 2\ell$ for some $\ell \in \mathbb{Z}$. Substituting into $n^2 = 2(2k-1)$, we get $(2\ell)^2 = 2(2k-1)$ and so $2^2\ell^2 = 2(2k-1)$, and dividing by 2 gives $2\ell^2 = 2k - 1$. Rearranging gives

$$1 = 2k - 2\ell^2 = 2(k - \ell^2)$$

implying $2 \mid 1$. Since $2 \nmid 1$, we have a contradiction. Therefore if $n \in \mathbb{Z}$, then $4 \nmid n^2 + 2$. ∎