Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[2 parts, 5 points each]** Prove the following using either a direct proof or proof by contrapositive.

   (a) Let $x, y \in \mathbb{R}$. If $x + y$ is irrational, then x is irrational or y is irrational.

We show the contrapositive: if x and y are rational, then $x+y$ is rational.

Suppose x and y are rational. By definition, we have $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some $a,b,c,d \in \mathbb{Z}$ with $b \neq 0$ and $d \neq 0$. We compute

$$x+y = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}.$$

Since $ad+bc$ and bd are integers, it follows that $x+y$ is rational. $\blacksquare$

   (b) Let a and b be positive integers. Prove that if $a - b > 1$, then $a^2 - b^2$ is not prime. (Hint: what algebraic formulas apply to a difference of squares?)

We give a direct proof. Let a and b be positive integers such that $a-b>1$. We show that a^2-b^2 is not prime. Indeed, $a^2-b^2 = (a-b)(a+b)$.

By assumption, $a-b>1$. Also $a+b \geq 1+1 \geq 2$ since a and b are positive integers. Since $a+b \geq 2$, it follows that $a^2-b^2 \geq (a-b)\cdot 2$, and so $a-b \leq \frac{1}{2}(a^2-b^2)$.

So $1 < a-b \leq \frac{1}{2}(a^2-b^2)$. Since $(a-b) \mid a^2-b^2$ and $a-b$ is not 1 or a^2-b^2, it follows that a^2-b^2 has a positive divisor other than 1 and itself. Therefore a^2-b^2 is not prime.