Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[4 points]** Suppose that $a, b, c \in \mathbb{Z}$. Prove that if $a \mid b$ and $a \mid c$, then $a \mid b + c$.

Since $a | b$ and $a | c$, we have $b = ka$ and $c = \ell a$ for some $k, \ell \in \mathbb{Z}$. We compute $b + c = ka + \ell a = (k+\ell)a$. Since $k+\ell$ is an integer, it follows that $a | b+c$. ∎

2. **[3 points]** Suppose that $x, y \in \mathbb{R}$. Prove that if $7x^2 + y = 7x + xy$, then $x = 1$ or $y = 7x$.

Suppose that $7x^2 + y = 7x + xy$. Rearranging these terms and simplifying, we have

$$0 = 7x^2 - 7x + y - xy = 7x(x-1) + y(1-x)$$
$$= 7x(x-1) - y(x-1) = (7x - y)(x-1).$$

Since a product only gives zero when one of its factors is zero, it follows that $x - 1 = 0$ or $7x - y = 0$. In the first case, $x = 1$ and in the second case, $y = 7x$. ∎

3. **[3 points]** Suppose that $n \in \mathbb{Z}$. Prove that if $3 \nmid n$, then $3 \mid n^2 + 2$. Hint: use the division lemma to divide n by 3 and try cases depending on the remainder.

Suppose that $3 \nmid n$. By the division lemma, there exist $q, r \in \mathbb{Z}$ such that $n = 3q + r$ and $0 \leq r \leq 2$. However, since $3 \nmid n$, we may rule out $r = 0$. Therefore $1 \leq r \leq 2$ and so $r = 1$ or $r = 2$.

Case 1: If $r = 1$, then we have $n^2 + 2 = (3q+1)^2 + 2 = 9q^2 + 6q + 3 = 3(q^2 + 2q + 1)$.
Since $q^2 + 2q + 1 \in \mathbb{Z}$, we have $3 \mid n^2 + 2$.

Case 2: If $r = 2$, then we have $n^2 + 2 = (3q+2)^2 + 2 = 9q^2 + 12q + 6 = 3(3q^2 + 4q + 2)$.
Since $3q^2 + 4q + 2 \in \mathbb{Z}$, we have $3 \mid n^2 + 2$.

In all cases, $3 \mid n^2 + 2$. ∎