Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[3 parts, 1 point each]** Translate the following sentences to symbolic logic as directly and simply as possible. Is the statement true or false? Write the entire word.

   (a) The emptyset is a subset of every set.

   $\boxed{\forall \text{ sets } S,\ \emptyset \subseteq S}$

   This is $\boxed{\text{true}}$. No matter what S is, every elt in $\emptyset$ is also an elt. in S.

   (b) There is an integer which is both a perfect square and a perfect cube.

   $\boxed{\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \exists z \in \mathbb{Z},\ x = y^2 \wedge x = z^2.}$

   $\boxed{\text{True}}$. For example $1 = 1^2$ and $1 = 1^3$.

   (c) Each rational number can be bounded above by some integer.

   $\boxed{\forall x \in \mathbb{Q}, \exists y \in \mathbb{Z},\ x \leq y.}$

   $\boxed{\text{True.}}$ Given $x \in \mathbb{Q}$, we can always round up to the next integer y.

2. **[3 parts, 1 point each]** Translate the following statements/open sentences in symbolic logic to English sentences as simply as possible. Is the statement true or false? Write the entire word.

   (a) $\sim (\exists x \in \mathbb{Q}, x^2 = 2)$

   The number $\sqrt{2}$ is not rational.

   This is $\boxed{\text{true}}$.

   (b) $\forall S \subseteq \mathbb{R}, [(\exists a \in \mathbb{N}, |S| \leq a) \implies (\exists m \in S, \forall x \in S, x \leq m)]$

   Every finite set of real numbers has a maximum element.

   This is $\boxed{\text{true}}$.

   (c) $\forall x \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}, [\varepsilon > 0 \implies \exists y \in \mathbb{Q}, |x - y| < \varepsilon]$

   Soln 1: For each real number x and each positive ε, there is a rational number y which is at distance at most ε from x.

   Soln 2: Each real number can be arbitrarily well approximated by a rational.

   This is $\boxed{\text{true}}$

3. **[2 parts, 2 points each]** Negate the following sentences as simply and naturally as possible. (You may translate to and from symbolic logic if helpful, but this is not required.) Is the **original statement** true or false? Explain.

   (a) Every pair of distinct lines in the plane intersects at a unique point.

$\forall$ lines ℓ_1, $\forall$ lines ℓ_2, $\ell_1 \neq \ell_2 \Rightarrow |\ell_1 \cap \ell_2| = 1.$

Negate: $\exists$ line ℓ_1, $\exists$ line ℓ_2, $\ell_1 \neq \ell_2 \wedge |\ell_1 \cap \ell_2| \neq 1.$

In English: There exists a pair of distinct lines which intersect in zero or more than 1 point.

The original statement is $\boxed{\text{false}}$: two distinct parallel lines do not intersect.

   (b) There is an infinite set of integers such that no integer in the set divides some other integer in the set.

$\exists S \subseteq \mathbb{Z}, \left[(S \text{ infinite}) \wedge \left(\forall x \in S, \forall y \in S, \ x \neq y \Rightarrow x \text{ does not divide } y \right) \right]$

Negate: $\forall S \subseteq \mathbb{Z} \left[(S \text{ is finite}) \vee \left(\exists x \in S, \exists y \in S, (x \neq y \wedge x \text{ divides } y) \right) \right]$

$\forall S \subseteq \mathbb{Z} \left[(S \text{ is infinite}) \Rightarrow \left(\exists x \in S, \exists y \in S, (x \neq y \wedge x \text{ divides } y) \right) \right]$

In English: Every infinite set of integers has a pair of distinct integers with one dividing the other.

The original statement is $\boxed{\text{true}}$. For example, if $S = \{p : p \text{ is prime}\}$, then no integer in S divides some other integer in S.