Directions: You may work to solve these problems in groups, but all written work must be your own. Unless the problem indicates otherwise, all problems require some justification; a correct answer without supporting reasoning is not sufficient. Submissions must be stapled. See "Guidelines and advice" on the course webpage for more information.

1. Use the binomial theorem to find the coefficient of x^4y^8 in $(3x-2y)^{12}$.

2. An identity.

   (a) Using the binomial theorem, show that $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n$. (Hint: substitute appropriate values for x and y in the equation from the binomial theorem.)

   (b) By counting the subsets of an n-element set in two different ways, give a combinatorial proof that $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n$.

3. Aces. Recall that a standard deck of cards has one card for each suit/rank pair, where the 4 suits are clubs, diamonds, hearts, and spades, and the 13 ranks are ace, 2 through 10, jack, queen, and king. A poker hand is a set of 5 cards. Use a sentence or two to explain in detail how your how you derived your answer. Express your answers first using binomial coefficients, and then apply the algebraic formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to obtain simplified numbers.

   (a) How many poker hands contain the ace of spades? If dealt a poker hand from a freshly shuffled deck, what is the probability it contains the ace of spades?

   (b) How many poker hands contain exactly one ace?

   (c) How many poker hands contain no aces?

   (d) How many poker hands contain at least two aces?

4. Prove the following (using any method).

   (a) Suppose that $x \in \mathbb{Z}$. Prove that x is even if and only if $3x+5$ is odd.

   (b) An integer a is odd if and only if a^3 is odd.

   (c) Suppose $x, y \in \mathbb{R}$. Then $(x+y)^2 = x^2+y^2$ if and only if $x=0$ or $y=0$.

5. Prove that for each $d \in \mathbb{N}$, there exists a prime p such that $p > 1000$ and $p+d$ is not prime. (Hint: try proof by contradiction. What does it mean for this claim to be false? In other words, what is the logical negation of this claim? If you have trouble formulating the negation, then it may be helpful to translate the claim into a symbolic logic formula φ, negate φ and simplify, and finally translate back into English.)