Name: ____________________

Directions: Show all work. No credit for answers without work. Except when asked for an explicit numerical answer, you may leave answers in terms of binomial/multinomial coefficients, factorials, and sums with a small number of terms.

1. [**9 points**] A committee of 5 people must be chosen from a group of 14 employees. How many ways can the committee be chosen? Give an explicit numerical answer.

2. [**2 parts, 8 points each**] A standard deck of cards has one card for each suit/rank pair, where the suits are spades, hearts, diamonds, and clubs, and the ranks are ace, 2 through 10, jack, queen, and king.

   (a) How many ways are there to choose a set of 5 cards from the deck with at least 2 clubs?

   (b) The cards are shuffled and dealt to 4 people, with each person receiving 13 cards. What is the probability that each person's hand has exactly one king?

3. **[3 parts, 6 points each]** How many ways are there to arrange the letters in the word ENTENTE:

   (a) without any restrictions?

   (b) so that the E's are all next to each other (as in NTEEENT)?

   (c) so that no two E's are consecutive?

4. **[3 parts, 6 points each]** Count the number of non-negative integer solutions to the following.

   (a) $x_1 + \ldots + x_6 = 30$

   (b) $x_1 + \ldots + x_6 = 30$, such that $x_i \geq i$ for $1 \leq i \leq 6$

   (c) $x_1 + \ldots + x_6 = 30$ such that $x_i \leq 20$ for each i.

5. [**10 points**] Give an algebraic and combinatorial proofs of the identity $t^3 = 6\binom{t}{3} + 6\binom{t}{2} + \binom{t}{1}$.

6. [**5 points**] Use the identity in the previous problem to give a formula for $\sum_{t=1}^{n} t^3$. (Hint: an identity from HW10 may be helpful; it counts the number of $(k+1)$-element subsets of $[n+1]$ by grouping the subsets by maximum value.)

7. **[8 points]** Use the binomial theorem to find the coefficient of x^7 in the expansion of $(x+1)^{20}$.

8. **[2 parts, 8 points each]** Find simple formulas for the following sums.

   (a) $\sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^k$

   (b) $\sum_{k=0}^{n} \binom{n}{k} k 2^k$ (Hint: differentiate the binomial theorem expansion for $(x+1)^n$.)