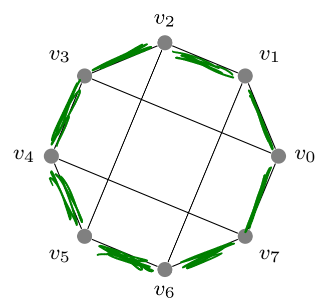
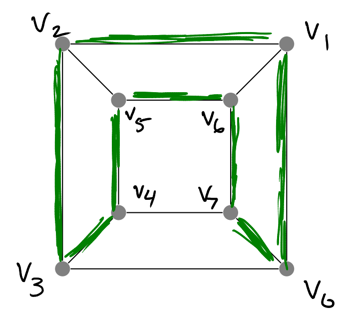
Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[15 points]** Are the following graphs isomorphic? Either give an isomorphism or explain why not.

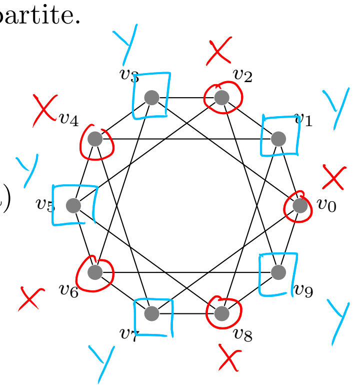




Yes, they are $\boxed{\text{isomorphic}}$. In addition to the edges of the green C_8 subgraph, we have chords v_0v_3, v_4v_7, v_1v_6, and v_2v_5.

2. **[2 parts, 5 points each]** For each of the following graphs, decide whether the graph is bipartite or not. If bipartite, then give a bipartition; otherwise, explain why the graph is not bipartite.

(a)

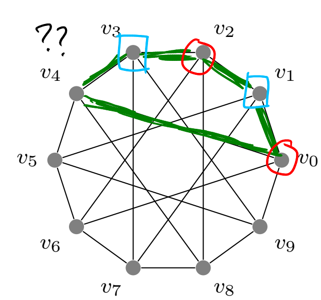


$\boxed{\text{Bipartite}}$ with bipartition

$X = \{v_0, v_2, v_4, v_6, v_8\}$

$Y = \{v_1, v_3, v_5, v_7, v_9\}$

(b)

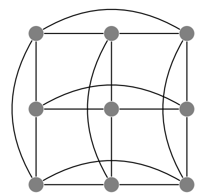


This graph has C_5 as a subgraph:

$v_0 v_1 v_2 v_3 v_4$

A bipartition would have to alternate parts along the cycle, which is not possible.

3. **[10 points]** Let G_n be the graph whose vertices are the lattice points of an $n \times n$ grid, where two points are adjacent if and only if they are in the same row or in the same column. A copy of G_3 appears below.



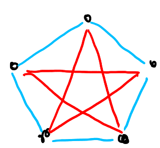
Find a formula for the number of edges in G_n.

Note: each vertex has degree $2(n-1)$

So $|E(G)| = \frac{1}{2}\sum_{v} \deg(v) = \frac{1}{2}\sum_{v} 2(n-1) = \frac{1}{\cancel{2}} \cdot n^2 \cdot \cancel{2}(n-1) = \boxed{n^2(n-1)}$

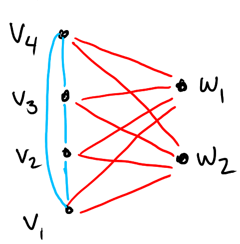
4. **[15 points]** Recall that $K_{1,3}$ is the complete bipartite graph with one singleton part and another part with 3 vertices. Prove that $r(K_{1,3}, C_4) = 6$. [Hint: use that $r(C_4, C_4) = 6$.]

First, we show $K_5 \not\to K_{1,3}, C_4$. We color K_5 with two monochromatic 5-cycles:



The blue subgraph has no copy of $K_{1,3}$ and the red subgraph has no copy of C_4. Therefore $r(K_{1,3}, C_4) > 5$.

Next, we show $K_6 \to K_{1,3}, C_4$. Let G be a blue/red edge-coloring of K_6 and suppose for a contradiction that G has no blue copy of $K_{1,3}$ and no red copy of C_4. Since $r(C_4, C_4) = 6$, it follows that G has a monochromatic copy of C_4, and it must be in blue. Let v_1, v_2, v_3, v_4 be the vertices of the blue 4-cycle,



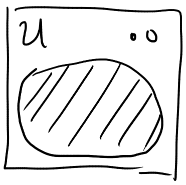
and let w_1, w_2 be the remaining vertices in G. If some edge $v_i w_j$ is blue, then $v_i w_j$ completes a blue copy of $K_{1,3}$ with center vertex v_i. So all edges of the form $v_i w_j$ are red. But now we have red copies of C_4, such as $w_1 v_1 w_2 v_2$. This is a contradiction and it follows that $r(K_{1,3}, C_4) \le 6$.

5. **[3 parts, 10 points each]** How many numbers in $\{1, \ldots, 700\}$:

(a) have only even digits?

note we allow zero

Let $\mathcal{U} = \{x : 0 \le x \le 700$ and all digits of x are even$\}$



_ _ _ [3 digits, leading zeros allowed]

1. Choose first digit from $\{0, 2, 4, 6\}$ $n_1 = 4$
2. Choose second digit from $\{0, 2, 4, 6, 8\}$ $n_2 = 5$
3. Choose third digit from $\{0, 2, 4, 6, 8\}$ $n_3 = 5$.

So $|\mathcal{U}| = 4 \cdot 5 \cdot 5 = 100$.

To get our set, we need to remove 0 (generated by $\underline{0}\ \underline{0}\ \underline{0}$). So $|\mathcal{U}| - 1 = \boxed{99}$.

(b) have 2 odd digits and 1 even digit?

Note: 700 and 000 both not allowed.

Addition Rule:

odd odd even	+ odd even odd	+ even odd odd
$\{1,3,5\}$ $\{1,3,5,7,9\}$ $\{0,2,4,6,8\}$	$\{1,3,5\}$ $\{0,2,4,6,8\}$ $\{1,3,5,7,9\}$	$\{2,4,6\}$ $\{1,3,5,7,9\}$ $\{1,3,5,7,9\}$
$3 \cdot 5 \cdot 5$	$3 \cdot 5 \cdot 5$	$3 \cdot 5 \cdot 5$

So total number is $3(3 \cdot 5^2) = 3^2 \cdot 5^2 = (15)^2 = \boxed{225}$

(c) have no repeated digits?

Use Addition Principle. Note: 700 not allowed

1-digit numbers	2-digit numbers	3-digit numbers
$\{1, \ldots, 9\}$	Stage 1: $\{1, \ldots, 9\}$ Stage 2: $\{0, \ldots, 9\}$ not same as Stage 1	Stage 1 $\{1, \ldots, 6\}$ Stage 2: $\{0, \ldots, 9\}$ Diff from Stage 1 Stage 3: $\{0, \ldots, 9\}$ Diff from Stage 1&2
9	$9 \cdot (10 - 1) = 81$	$6 \cdot (10-1) \cdot (10-2)$ $6 \cdot 9 \cdot 8 = 432$

So total number is $9 + 81 + 432 = \boxed{522}$

6. Suppose we roll a 6-sided die n times. Let A_n be the event that we do not roll a pair of 6's twice in a row.

   (a) **[15 points]** Let $f(n) = |A_n|$. Find a recurrence relation for $f(n)$. Include necessary base cases.

Note: Sample space $S = [6]^n$ and $A_n = \{(x_1, \dots, x_n) \in S : (x_1, \dots, x_n) \text{ has no consecutive 6's}\}$.

$f(0) = 1$, $f(1) = 6$. Suppose $n \geq 2$. Let B_n be the sequences in A_n that end with a non-6 value and let C_n be the sequences in A_n that end in 6. Sequences in B_n

$$\overline{x_1} \quad \cdots \quad \overline{x_{n-2}} \quad \overline{x_{n-1}} \quad \overline{x_n}$$

have last entry in $\{1, \dots, 5\}$ appended to a sequence in A_{n-1}. Moreover, appending a value from $\{1, \dots, 5\}$ to a sequence in A_{n-1} gives a sequence in B_n. So $|B_n| = 5|A_{n-1}| = 5f(n-1)$. Similarly a sequence in C_n has a last value of 6 appended to a sequence in B_{n-1}. So $|C_n| = |B_{n-1}| = 5|A_{n-2}| = 5f(n-2)$. We have $f(n) = |A_n| + |B_n|$
$= 5f(n-1) + 5f(n-2)$ for $n \geq 2$.

   (b) **[5 points]** Use your recurrence in part (a) to find the probability of not getting 6's twice in a row when rolling a die 4 times.

From part (a),
$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 6 & \text{if } n = 1 \\ 5(f(n-1) + f(n-2)) & \text{if } n \geq 2 \end{cases}$$

n	0	1	2	3	4
$f(n)$	1	6	35	205	1200
			$5(6+1)$	$5(35+6)$	$5(205+35) = 5(240)$

So probability is $\dfrac{|A_4|}{|S|} = \dfrac{f(4)}{6^4} = \dfrac{1200}{6^4} = \dfrac{12 \cdot (10)^2}{6^4} = \dfrac{2 \cdot (10)^2}{6^3} = \dfrac{2 \cdot 2^2 \cdot 5^2}{6^3}$

$$= \frac{\cancel{2^3} \cdot 5^2}{\cancel{2^3} \cdot 3^3} = \boxed{\frac{25}{27}} \approx 0.9259$$