Name: ______________________________

Directions: Show all work. No credit for answers without work.

1. [**18 points**] Prove that if $n \geq 0$, then $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

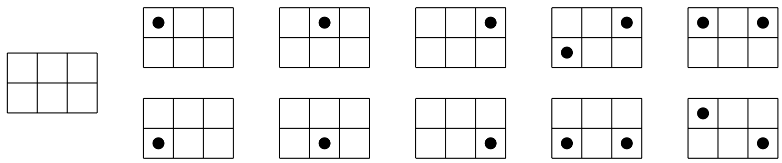
2. **[18 points]** Prove that $\sum_{k=1}^{n} 2^{k-1}3^{n-k} = 3^n - 2^n$ for $n \geq 0$ using the no minimum counter-example method.

3. Let $a_1 = \frac{1}{2}$ and $a_n = \frac{1}{2-a_{n-1}}$ for $n \geq 2$.

   (a) **[14 points]** Compute a_n for $n \leq 4$. Guess a formula for a_n.

   (b) **[18 points]** Prove that your formula is correct.

4. **[14 points]** For $n \geq 1$, let b_n be the number of ways to mark zero or more cells of a $(2 \times n)$-grid so that no two marked cells are next to each other vertically, horizontally, or diagonally. For example, $b_3 = 11$, as shown below.



Give a recurrence relation for b_n, complete with all necessary base cases. (No need to guess a formula for b_n or solve.)

5. **[18 points]** Let S be a subset of $\{1, \ldots, n\}$ with $|S| = m$. Prove that if $m > 1 + (n/2)$, then there exist distinct $x, y, z \in S$ such that $x + y = z$. (Hint: let $S = \{a_1, \ldots, a_m\}$ with $a_1 < \cdots < a_m$, and let $k = a_1$, the smallest integer in S. Consider the list $a_2, \ldots, a_m, b_2, \ldots, b_m$, where $b_i = a_i + k$.)