Name: Solutions

Directions: Show all work. No credit for answers without work.

1. [**18 points**] Prove that if $n \geq 0$, then $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

By induction on n. If $n=0$, then the LHS is an empty sum, giving 0, and the RHS is $\frac{0(0+1)}{2}$, which also equals 0.

Suppose $n \geq 1$. By the induction hypothesis, we have

$$\sum_{k=1}^{n-1} k = \frac{(n-1)n}{2}.$$

Adding n to both sides gives

$$\sum_{k=1}^{n} k = \frac{(n-1)n}{2} + n = \frac{(n-1)n + 2n}{2} = \frac{n(n+1)}{2}$$

and so the identity holds at n. $\boxtimes$

2. [**18 points**] Prove that $\sum_{k=1}^{n} 2^{k-1}3^{n-k} = 3^n - 2^n$ for $n \geq 0$ using the no minimum counter-example method.

Suppose for a contradiction that the claim is false, and let n be the least non-negative integer for which the claim fails. Since

$$\sum_{k=1}^{0} 2^{k-1}3^{n-k} = 0 = 3^0 - 2^0$$

we have $n \geq 1$. Since $n-1 \geq 0$ and n is the min. integer for which the claim fails, it follows that the claim holds at $n-1$:

$$\sum_{k=1}^{n-1} 2^{k-1}3^{(n-1)-k} = 3^{n-1} - 2^{n-1} \qquad (*).$$

Multiplying both sides of $(*)$ by 3 gives $\sum_{k=1}^{n-1} 2^{k-1}3^{n-k} = 3^n - 3\cdot 2^{n-1}$ and adding the $k=n$ term $2^{n-1}\cdot 3^{n-n}$ or 2^{n-1} to both sides gives

$$\sum_{k=1}^{n} 2^{k-1}3^{n-k} = 3^n - 3\cdot 2^{n-1} + 2^{n-1} = 3^n - 2\cdot 2^{n-1} = 3^n - 2^n$$

Therefore the claim holds at n afterall, and we have a contradiction. $\blacksquare$

3. Let $a_1 = \frac{1}{2}$ and $a_n = \frac{1}{2-a_{n-1}}$ for $n \geq 2$.

   (a) **[14 points]** Compute a_n for $n \leq 4$. Guess a formula for a_n.

$a_1 = \frac{1}{2}$

$a_2 = \frac{1}{2-\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$

$a_3 = \frac{1}{2-\frac{2}{3}} = \frac{3}{3\cdot 2-2} = \frac{3}{6-2} = \frac{3}{4}$

$a_4 = \frac{1}{2-\frac{3}{4}} = \frac{4}{4\cdot 2-3} = \frac{4}{8-3} = \frac{4}{5}$

Guess: $a_n = \frac{n}{n+1}$.

   (b) **[18 points]** Prove that your formula is correct.

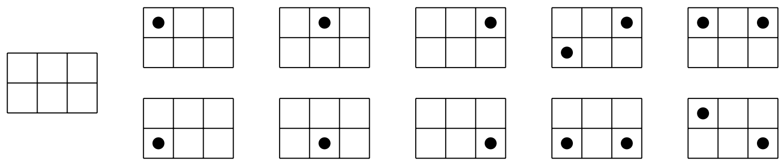
We prove $a_n = \frac{n}{n+1}$ by induction on n. If $n=1$, then $a_1 = \frac{1}{2}$ from the base case of the recurrence and $\frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$, and so the formula holds.

Suppose that $n \geq 2$. By definition, we have $a_n = \frac{1}{2-a_{n-1}}$. By the IH we have $a_{n-1} = \frac{n-1}{(n-1)+1} = \frac{n-1}{n}$. Therefore

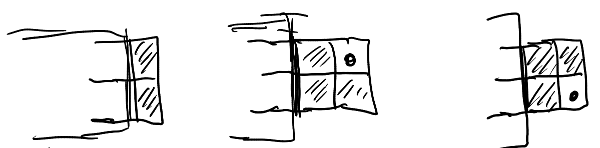
$$a_n = \frac{1}{2-a_{n-1}} = \frac{1}{2-\frac{n-1}{n}} = \frac{n}{2n-(n-1)} = \frac{n}{2n-n+1} = \frac{n}{n+1}$$

and so the formula for a_n also holds at n.

4. **[14 points]** For $n \geq 1$, let b_n be the number of ways to mark zero or more cells of a $(2 \times n)$-grid so that no two marked cells are next to each other vertically, horizontally, or diagonally. For example, $b_3 = 11$, as shown below.

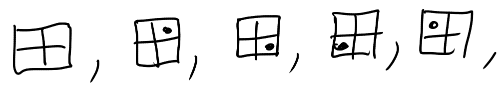


Give a recurrence relation for b_n, complete with all necessary base cases. (No need to guess a formula for b_n or solve.)

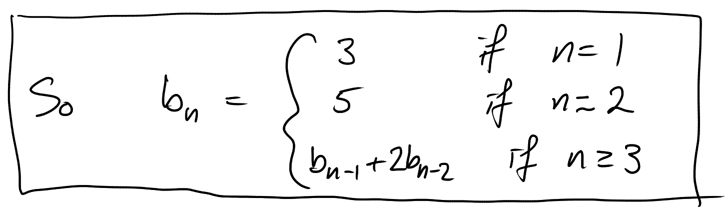


b_{n-1} $\quad$ b_{n-2} $\quad$ b_{n-2}

b_1:  $b_1 = 3$

b_2:  $b_2 = 5$

For $n \geq 3$, $b_n = b_{n-1} + 2b_{n-2}$.



So $\boxed{b_n = \begin{cases} 3 & \text{if } n = 1 \\ 5 & \text{if } n = 2 \\ b_{n-1} + 2b_{n-2} & \text{if } n \geq 3 \end{cases}}$

5. **[18 points]** Let S be a subset of $\{1, \ldots, n\}$ with $|S| = m$. Prove that if $m > 1 + (n/2)$, then there exist distinct $x, y, z \in S$ such that $x + y = z$. (Hint: let $S = \{a_1, \ldots, a_m\}$ with $a_1 < \cdots < a_m$, and let $k = a_1$, the smallest integer in S. Consider the list $a_2, \ldots, a_m, b_2, \ldots, b_m$, where $b_i = a_i + k$.)

Let $S = \{a_1, \ldots, a_m\}$ with $a_1 < \cdots < a_m$ and $k = a_1$ as suggested. We set $b_i = a_i + k$ for $2 \le i \le m$. Note that

$$k+1 \le a_2, \ldots, a_m, b_2, \ldots, b_m \le n+k$$

and so $a_2, \ldots, a_m, b_2, \ldots, b_m$ is a list of $2(m-1)$ integers in a range of size $(n+k)-k$ or n. Since $2(m-1) > 2\left(\frac{n}{2}\right) = n$, it follows that two distinct entries in the list are the same. It is impossible for $a_i = a_j$ with $i \ne j$, and also impossible for $b_i = b_j$ with $i \ne j$ since $b_2 < \cdots < b_m$. Therefore $a_i = b_j$ for distinct i and j.

We have $a_i = b_j = a_j + k = a_j + a_1$ with $1 < j < i$. Hence we have the desired $x, y, z \in S$ with $x = a_1$, $y = a_j$, and $z = a_i$. $\boxtimes$