Name: Solutions

**Directions:** Show all work.

1. **[5 points]** Let $a_0 = 2$ and $a_n = 2a_{n-1} - n$ for $n \geq 1$. Guess a formula for $a_n$ and prove your formula is correct.

First few values:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $a_n$ | 2 | $2\cdot2-1$ 3 | $2\cdot3-2$ 4 | $2\cdot4-3$ 5 | $2\cdot5-4$ 6 | $2\cdot6-5$ 7 |

Guess: $a_n = n+2$.

We prove $a_n = n+2$ by induction on $n$. If $n=0$, then $a_0 = 2$ and $n+2$ also evaluates to 2, so the formula holds at $n=0$.

Suppose $n \geq 1$. By definition, $a_n = 2a_{n-1} - n$. By the inductive hypothesis, we have $a_{n-1} = (n-1)+2 = n+1$. Substituting for $a_{n-1}$ gives

$$a_n = 2a_{n-1} - n = 2(n+1) - n = 2n+2-n = n+2$$

and it follows that the formula holds at $n$ also.

2. **[5 points]** Let $b_0 = 3$ and $b_n = 2b_{n-1} - n$ for $n \geq 1$. Prove that $b_n = 2^n + n + 2$.

By induction on $n$. If $n=0$, then $b_0 = 3$ and $2^n + n + 2 = 1+0+2 = 3$, and so the formula holds at $n=0$. Suppose $n \geq 1$. By definition we have $b_n = 2b_{n-1} - n$ and by I.H. we have $b_{n-1} = 2^{n-1} + (n-1) + 2$.

Combining these gives

$$b_n = 2b_{n-1} - n = 2\left[2^{n-1} + (n-1) + 2\right] - n$$

$$= 2\left[2^{n-1} + n + 1\right] - n$$

$$= 2^n + 2n + 2 - n$$

$$= 2^n + n + 2$$

and therefore the formula holds at $n$ also. ∎